
Transform Coding

- Principle of block-wise transform coding
- Properties of orthonormal transforms
- Discrete cosine transform (DCT)
- Bit allocation for transform coefficients
- Entropy coding of transform coefficients
- Typical coding artifacts
- Fast implementation of the DCT

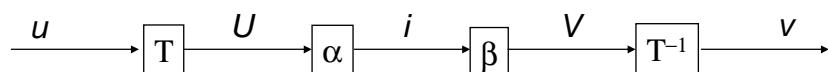


Transform Coding Principle

- Structure



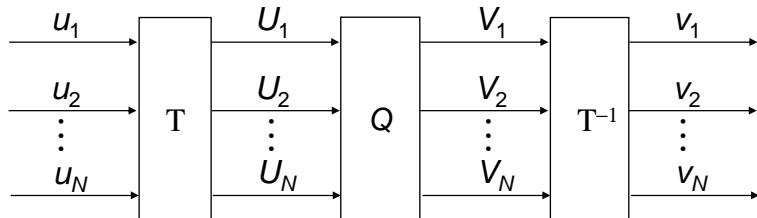
- Transform coder ($T\alpha$) / decoder (βT^{-1}) structure



- Insert entropy coding (γ) and transmission channel



Transform Coding and Quantization



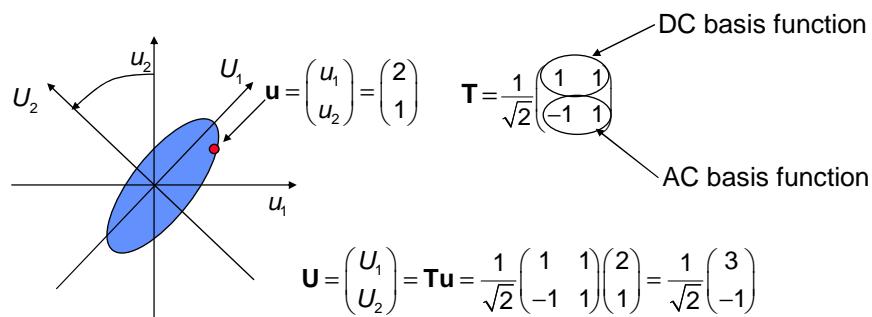
- Transformation of vector $\mathbf{u} = (u_1, u_2, \dots, u_N)$ into $\mathbf{U} = (U_1, U_2, \dots, U_N)$
- Quantizer Q may be applied to coefficients U_i
 - separately (scalar quantization: low complexity)
 - jointly (vector quantization, may require high complexity, exploiting of redundancy between U_i)
- Inverse transformation of vector $\mathbf{V} = (V_1, V_2, \dots, V_N)$ into $\mathbf{v} = (v_1, v_2, \dots, v_N)$

Why should we use a transform ?



Geometrical Interpretation

- A linear transform can decorrelate random variables
- An orthonormal transform is a rotation of the signal vector around the origin
- Parseval's Theorem holds for orthonormal transforms



Properties of Orthonormal Transforms

- Forward transform

$$\mathbf{U} = \mathbf{T}\mathbf{u}$$

N transform coefficients input signal block of size N
Transform matrix of size NxN

- Orthonormal transform property: inverse transform

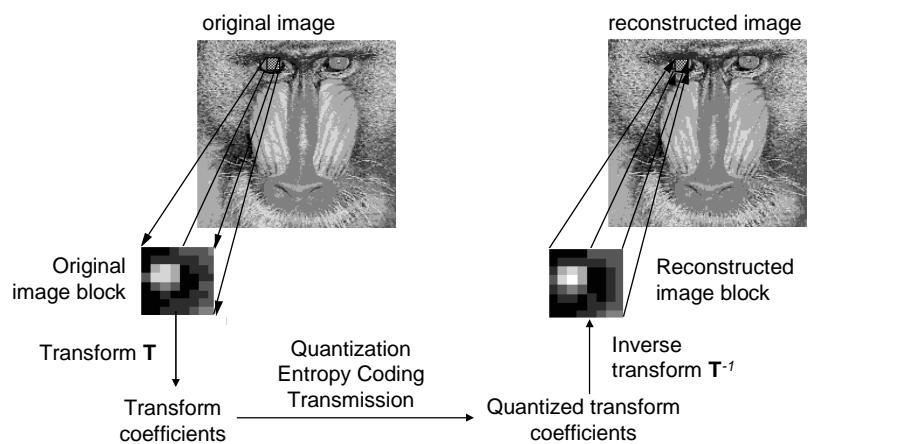
$$\mathbf{u} = \mathbf{T}^{-1}\mathbf{U} = \mathbf{T}^T\mathbf{U}$$

- Linearity: \mathbf{u} is represented as linear combination of “basis functions”



Transform Coding of Images

Exploit horizontal and vertical dependencies by processing blocks



Separable Orthonormal Transforms, I

- Problem: size of vectors $N \times N$ (typical value of N : 8)
- An orthonormal transform is separable, if the transform of a signal block of size $N \times N$ can be expressed by

$$\boxed{U = T u T^{-1}}$$

$\xrightarrow{\text{N} \times \text{N} \text{ transform coefficients}}$ $\xrightarrow{\text{Orthonormal transform matrix of size } N \times N}$ $\xrightarrow{\text{N} \times \text{N} \text{ block of input signal}}$

- The inverse transform is

$$\boxed{u = T^T U T}$$

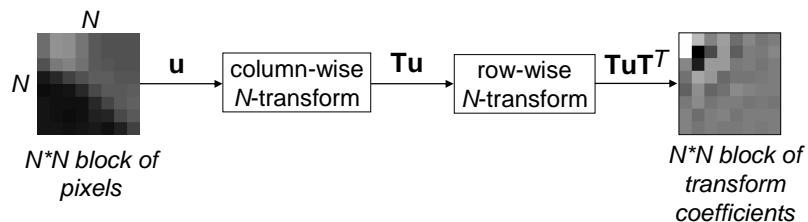
- Great practical importance: transform requires 2 matrix multiplications of size $N \times N$ instead one multiplication of a vector of size $1 \times N^2$ with a matrix of size $N^2 \times N^2$
- Reduction of the complexity from $O(N^4)$ to $O(N^3)$



Separable Orthonormal Transforms, II

Separable 2-D transform is realized by two 1-D transforms

- along rows and
- columns of the signal block



Criteria for the Selection of a Particular Transform

- Decorrelation, energy concentration
 - KLT, DCT, ...
 - Transform should provide energy compaction
- Visually pleasant basis functions
 - pseudo-random-noise, m-sequences, lapped transforms, ...
 - Quantization errors make basis functions visible
- Low complexity of computation
 - Separability in 2-D
 - Simple quantization of transform coefficients



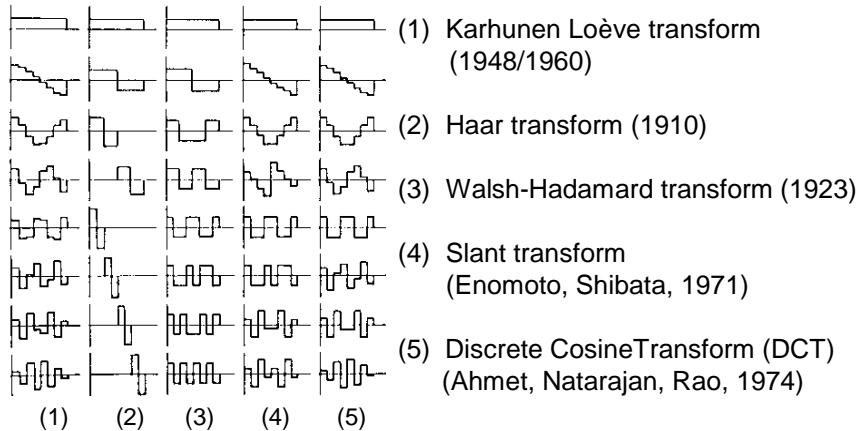
Karhunen Loève Transform (KLT)

- Decorrelate elements of vector \mathbf{u}
$$\mathbf{R}_u = E\{\mathbf{u}\mathbf{u}^T\}, \Rightarrow \mathbf{U} = \mathbf{T}\mathbf{u}, \Rightarrow \mathbf{R}_u = E\{\mathbf{U}\mathbf{U}^T\} = \mathbf{T}E\{\mathbf{u}\mathbf{u}^T\}\mathbf{T}^T = \mathbf{T}\mathbf{R}_u\mathbf{T}^T = \text{diag}\{\alpha_i\}$$
- Basis functions are eigenvectors of the covariance matrix of the input signal.
- KLT achieves optimum energy concentration.
- Disadvantages:
 - KLT dependent on signal statistics
 - KLT not separable for image blocks
 - Transform matrix cannot be factored into sparse matrices.



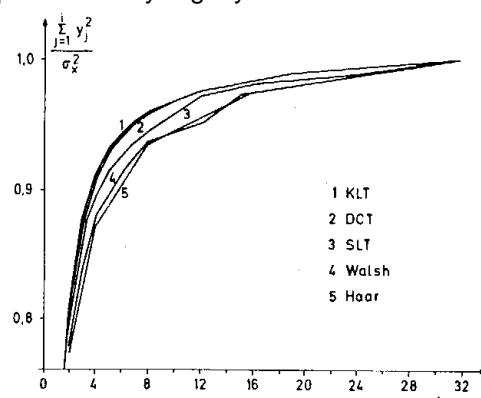
Comparison of Various Transforms, I

Comparison of 1D basis functions for block size $N=8$



Comparison of Various Transforms, II

- Energy concentration measured for typical natural images, block size 1x32 (Lohscheller)
- KLT is optimum
- DCT performs only slightly worse than KLT



DCT

- Type II-DCT of blocksize $M \times M$ is defined by transform matrix A containing elements

$$a_{ik} = \alpha_i \cdot \cos \frac{\pi(2k+1)i}{2M}$$

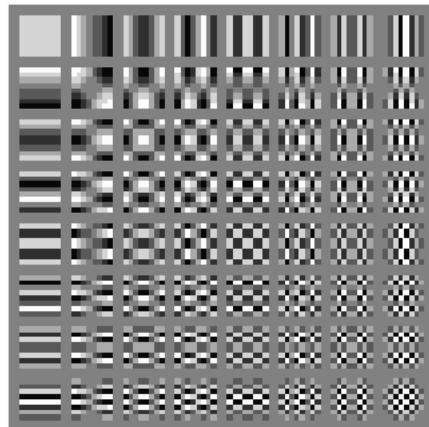
$i, k = 0 \dots (M-1)$

with

$$\alpha_0 = \sqrt{\frac{1}{M}}$$

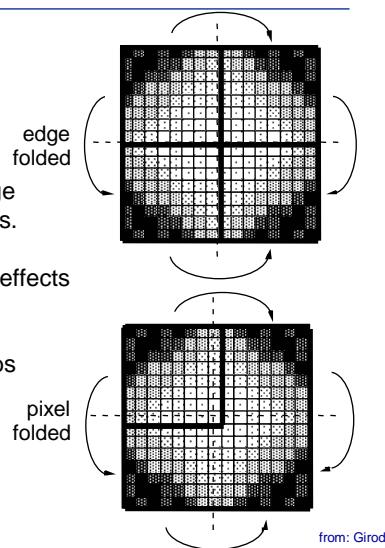
$$\alpha_i = \sqrt{\frac{2}{M}} \quad i \neq 0$$

- 2D basis functions of the DCT:



Discrete Cosine Transform and Discrete Fourier Transform

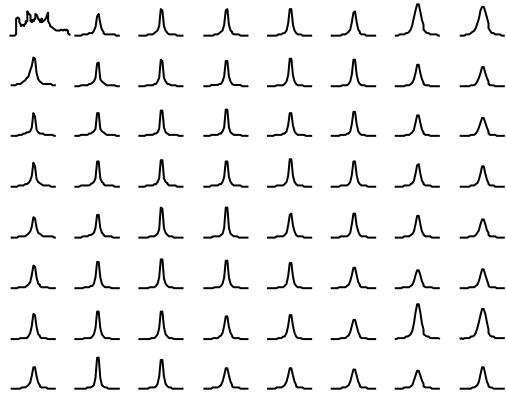
- Transform coding of images using the Discrete Fourier Transform (DFT):
- For stationary image statistics, the energy concentration properties of the DFT converge against those of the KLT for large block sizes.
- Problem of blockwise DFT coding: blocking effects
- DFT of larger symmetric block \rightarrow "DCT" due to circular topology of the DFT and Gibbs phenomena.
- Remedy: reflect image at block boundaries



Histograms of DCT Coefficients:



- Image: Lena, 256x256 pixel
- DCT: 8x8 pixels
- DCT coefficients are approximately distributed like Laplacian pdf



Distribution of the DCT Coefficients

- Central Limit Theorem requests DCT coefficients to be Gaussian distributed
- Model variance of Gaussian DCT coefficients distribution as random variable
(Lam & Goodmann'2000)

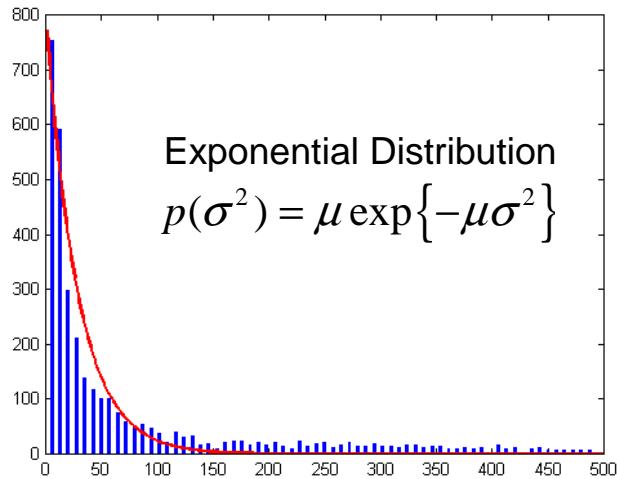
$$p(u | \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{u^2}{2\sigma^2}\right\}$$

- Using conditional probability

$$p(u) = \int_0^{\infty} p(u | \sigma^2) \cdot p(\sigma^2) \cdot d\sigma^2$$



Distribution of the Variance



Distribution of the DCT Coefficients

$$\begin{aligned} p(u) &= \int_0^\infty p(u | \sigma^2) \cdot p(\sigma^2) \cdot d\sigma^2 \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{u^2}{2\sigma^2}\right\} \cdot \mu \exp\{-\mu\sigma^2\} \cdot d\sigma^2 \\ &= \sqrt{\frac{2}{\pi}} \mu \int_0^\infty \exp\left\{-\frac{u^2}{2\sigma^2} - \mu\sigma^2\right\} \cdot d\sigma \\ &= \left(\sqrt{\frac{2}{\pi}} \mu\right) \left(\frac{1}{2} \sqrt{\frac{\pi}{\mu}}\right) \exp\left\{-2\sqrt{\frac{\mu u}{2}}\right\} \\ &= \frac{\sqrt{2\mu}}{2} \exp\left\{-\sqrt{2\mu}|u|\right\} \quad \text{Laplacian Distribution} \end{aligned}$$



Bit Allocation for Transform Coefficients I

- *Problem:* divide bit-rate R among N transform coefficients such that resulting distortion D is minimized.

$$D(R) = \frac{1}{N} \sum_{i=1}^N D_i(R_i), \quad \text{s.t.} \quad \frac{1}{N} \sum_{i=1}^N R_i \leq R$$

Average distortion Distortion contributed by coefficient i Rate for coefficient i Average rate

- *Approach:* minimize Lagrangian cost function

$$\frac{d}{dR_i} \sum_{i=1}^N D_i(R_i) + \lambda \sum_{i=1}^N R_i = \frac{dD_i(R_i)}{dR_i} + \lambda = 0$$

- *Solution:* Pareto condition

$$\boxed{\frac{dD_i(R_i)}{dR_i} = -\lambda}$$

- Move bits from coefficient with small distortion reduction per bit to coefficient with larger distortion reduction per bit



Bit Allocation for Transform Coefficients II

- *Assumption:* high rate approximations are valid

$$D_i(R_i) \approx a\sigma_i^2 2^{-2R_i}, \quad \rightarrow \quad \frac{dD_i(R_i)}{dR_i} \approx -2a\ln 2\sigma_i^2 2^{-2R_i} = -\lambda$$

$$R_i \approx \log_2 \sigma_i + \log_2 \sqrt{\frac{2a\ln 2}{\lambda}} \quad \rightarrow \quad R_i \approx \log_2 \sigma_i - \log_2 \tilde{\sigma} + R = \log_2 \frac{\sigma_i}{\tilde{\sigma}} + R$$

$$R = \frac{1}{N} \sum_{i=1}^N R_i = \underbrace{\frac{1}{N} \sum_{i=1}^N \log_2 \sigma_i}_{\log_2 \tilde{\sigma}} + \log_2 \sqrt{\frac{2a\ln 2}{\lambda}} = \underbrace{\log_2 \left(\prod_{i=1}^N \sigma_i \right)^{\frac{1}{N}}}_{\tilde{\sigma}} + \log_2 \sqrt{\frac{2a\ln 2}{\lambda}}$$

- Operational Distortion Rate function for transform coding:

$$D(R) = \frac{1}{N} \sum_{i=1}^N D_i(R_i) \approx \frac{a}{N} \sum_{i=1}^N \sigma_i^2 2^{-2R_i} = \frac{a}{N} \sum_{i=1}^N \sigma_i^2 2^{-2\log_2 \frac{\sigma_i}{\tilde{\sigma}} - 2R} = \boxed{a\tilde{\sigma}^2 2^{-2R}}$$

Geometric mean: $\tilde{\sigma} = \left(\prod_{i=1}^N \sigma_i \right)^{\frac{1}{N}}$



Entropy Coding of Transform Coefficients

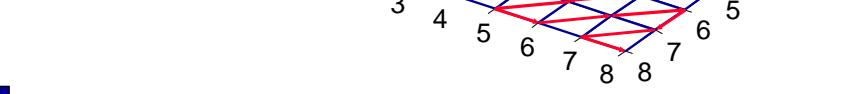
- Previous derivation assumes: $R_i = \frac{1}{2} \max[(\log_2 \frac{\sigma_i^2}{D}), 0] \text{ bit}$

- AC coefficients are very likely to be zero

Ordering of the transform coefficients by zig-zag scan

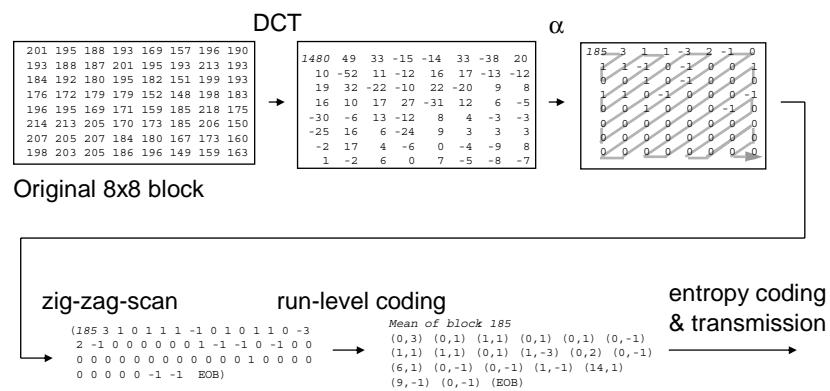
- Design Huffman code for event:
of zeros and coefficient value
- Arithmetic code maybe simpler

Probability that coefficient is not zero when quantizing with:
 $V_i = 100 * \text{round}(U/100)$

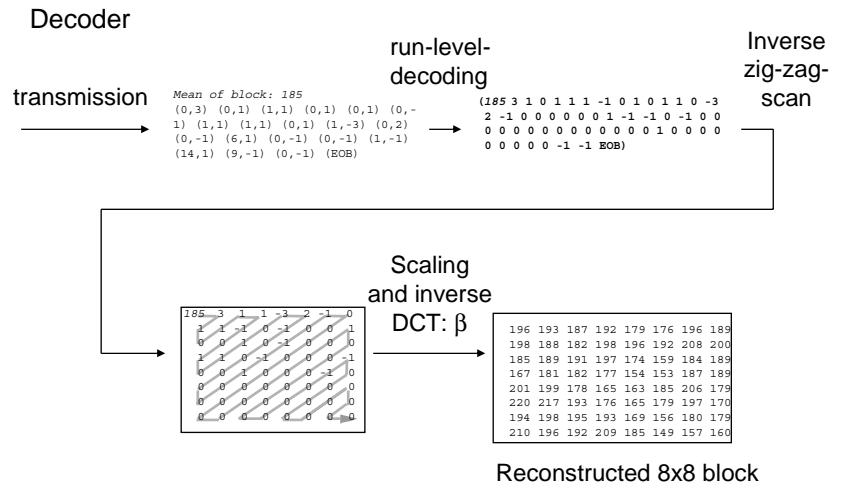


Entropy Coding of Transform Coefficients II

Encoder



Entropy Coding of Transform Coefficients III

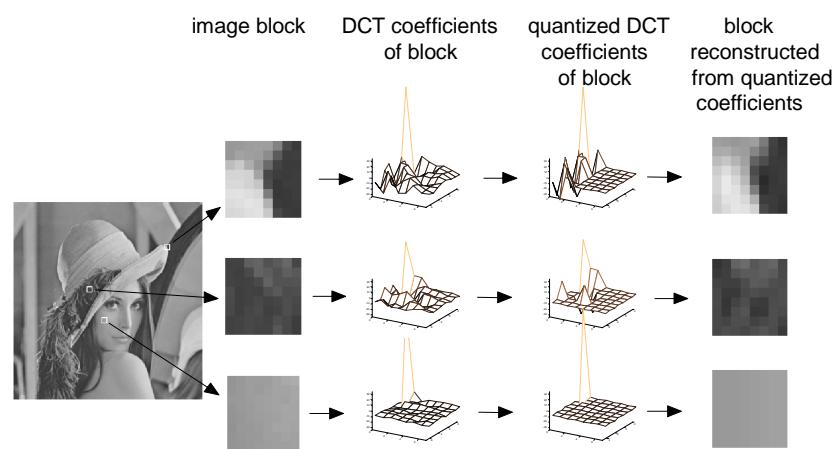


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Detail in a Block vs. DCT Coefficients Transmitted



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Typical DCT Coding Artifacts

DCT coding with increasingly coarse quantization, block size 8x8



quantizer step size
for AC coefficients: 25

quantizer step size
for AC coefficients: 100

quantizer step size
for AC coefficients: 200

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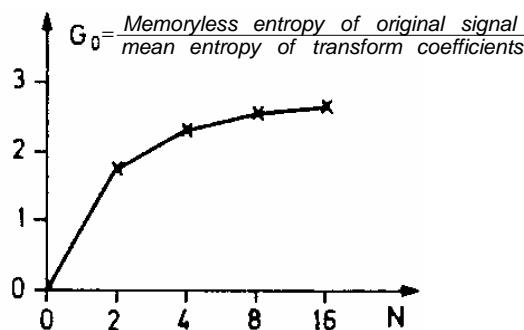


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Influence of DCT Block Size

- Efficiency as a function of block size $N \times N$, measured for 8 bit Quantization in the original domain and equivalent quantization in the transform domain.



- Block size 8x8 is a good compromise between coding efficiency and complexity



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Fast DCT Algorithm I

DCT matrix factored into sparse matrices (Arai, Agui, and Nakajima; 1988):

$$y = \underline{M} \cdot \underline{x}$$

$$= \underline{S} \cdot \underline{P} \cdot \underline{M}_1 \cdot \underline{M}_2 \cdot \underline{M}_3 \cdot \underline{M}_4 \cdot \underline{M}_5 \cdot \underline{M}_6 \cdot \underline{x}$$

$$\underline{S} = \begin{bmatrix} s_0 & & & \\ s_1 & s_2 & 0 & \\ s_3 & s_4 & & \\ s_5 & s_6 & & \\ 0 & s_7 & & \end{bmatrix} \quad \underline{P} = \begin{bmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \\ 1 & & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} \quad \underline{M}_1 = \begin{bmatrix} 1 & & & \\ & 1 & 0 & \\ & & 1 & 1 \\ & & & 1 \\ 0 & & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} \quad \underline{M}_2 = \begin{bmatrix} 1 & & & \\ & 1 & 1 & 0 \\ & & -1 & 1 \\ & & & 1 \\ 0 & & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$$

$$\underline{M}_3 = \begin{bmatrix} 1 & & & \\ & 1 & 0 & \\ & & 1 & -1 \\ & & & 0 \\ 0 & & & \\ & -1 & 1 & \\ & & -1 & 1 \\ & & & 1 \end{bmatrix} \quad \underline{M}_4 = \begin{bmatrix} 1 & 1 & & 0 \\ 1 & -1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ 0 & & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} \quad \underline{M}_5 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & \\ & -1 & 1 & \\ & & 1 & 1 \\ 0 & & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} \quad \underline{M}_6 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & \\ & 1 & 1 & 0 \\ & & 1 & 1 \\ 0 & & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$$

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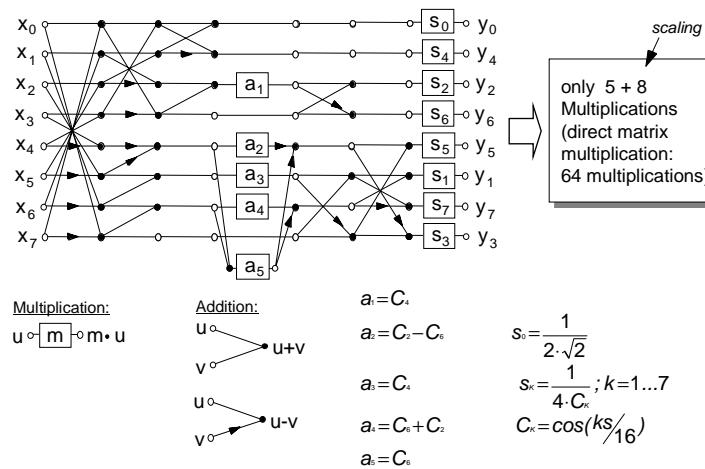


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Fast DCT Algorithm II

Signal flow graph for fast (scaled) 8-DCT according to Arai, Agui, Nakajima:



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Transform Coding: Summary

- Orthonormal transform: rotation of coordinate system in signal space
- Purpose of transform: decorrelation, energy concentration
- KLT is optimum, but signal dependent and, hence, without a fast algorithm
- DCT shows reduced blocking artifacts compared to DFT
- Bit allocation proportional to logarithm of variance
- Threshold coding + zig-zag scan + 8x8 block size is widely used today (e.g. JPEG, MPEG-1/2/4, ITU-T H.261/2/3)
- Fast algorithm for scaled 8-DCT: 5 multiplications, 29 additions

