# Rate Distortion Theory & Quantization

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- *R*(*D**) for Memoryless Gaussian Sources
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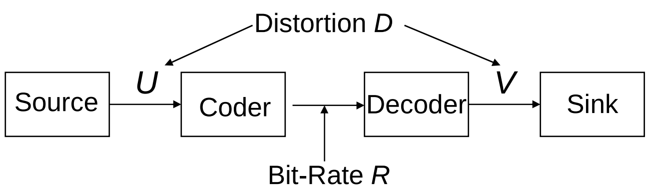
# Rate Distortion Theory

- Theoretical discipline treating data compression from the viewpoint of information theory.

- Rate distortion theory calculates minimum transmission bit-rate *R* for a given distortion *D*.




- Results of rate distortion theory are obtained without consideration of a specific coding method.

## Rate Distortion Theory: Mutual Information

- "Mutual information" is the information that symbols $u$ and symbols $v$ convey about each other.

$$\boxed{I(u_l;v_k)=I(v_k)-I(v_k\,|\,u_l)=\log\frac{P(v_k\,|\,u_l)}{P(v_k)}=\log\frac{P(u_l\,|\,v_k)}{P(u_l)}}$$

- Average mutual information:

$$\begin{aligned}I(U;V)&=\sum_u\sum_v P(u,v)\cdot\log\frac{P(v\,|\,u)}{P(v)}=\sum_u\sum_v P(u,v)\cdot\log\frac{P(u\,|\,v)}{P(u)}\\&=H(V)-H(V\,|\,U)=H(U)-H(U\,|\,V)\end{aligned}$$

- Properties of mutual information:

$$\begin{gathered}0\le I(U;V)=I(V;U)\\I(U;V)\le H(U)\\I(V;U)\le H(V)\end{gathered}$$



## Rate Distortion Theory: Distortion

- Symbol $u$ sent, $v$ received
- Distortion:

$$\begin{gathered}d(u,v)\ge 0\\d(u,v)=0 \quad \text{for} \quad u=v\end{gathered}$$

- Average distortion:

$$\boxed{D=E[d(u,v)]=\sum_u\sum_v P(v\,|\,u)\cdot P(u)\cdot d(u,v)}$$

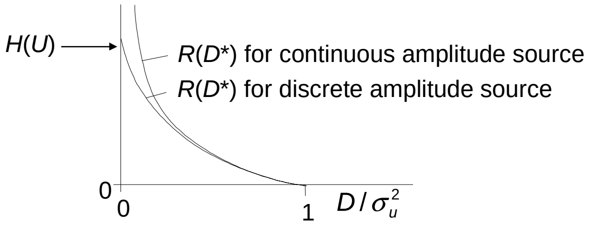
- Distortion criterion: $\boxed{D\le D^*}$

($D^*$ -maximum average distortion)

# Rate Distortion Function

- Definition: $$\boxed{R(D^*) = \min_{D \le D^*} \{I(U;V)\}}$$

- For a given maximum average distortion $D^*$, the rate distortion function $R(D^*)$ is the lower bound for the transmission bit-rate.






# Shannon Lower Bound

- It can be shown that $\boxed{H(U-V\,|\,V) = H(U\,|\,V)}$

- Then we can write
$$\boxed{\begin{aligned} R(D^*) &= \min_{D \le D^*} \{H(U) - H(U\,|\,V)\} \\ &= H(U) - \max_{D \le D^*} \{H(U\,|\,V)\} \\ &= H(U) - \max_{D \le D^*} \{H(U-V\,|\,V)\} \end{aligned}}$$

- Ideally, the source coder would produce distortions *u*-*v* that are statistically independent from the reconstructed signal *v* (not always possible!).

- Shannon Lower Bound: $$\boxed{R(D^*) \ge H(U) - \max_{D \le D^*} H(U-V)}$$

## $R(D^*)$ for a Memoryless Gaussian Source and MSE Distortion

- Gaussian source, variance $\sigma^2$
- Mean squared error (MSE) $D = E\{(u-v)^2\}$

$$R(D^*) = \frac{1}{2}\log\frac{\sigma^2}{D^*};\ D(R^*) = \sigma^2 2^{-2R^*}, R \geq 0$$

$$SNR = 10\cdot\log_{10}\frac{\sigma^2}{D} = 10\cdot\log_{10} 2^{-2R} \approx 6R\ [dB]$$

- Rule of thumb: 6 dB ~ 1 bit
- The $R(D^*)$ for non-Gaussian sources with the same variance $\sigma^2$ is always below this Gaussian $R(D^*)$ - curve.



## $R(D^*)$ Function for Gaussian Source with Memory I

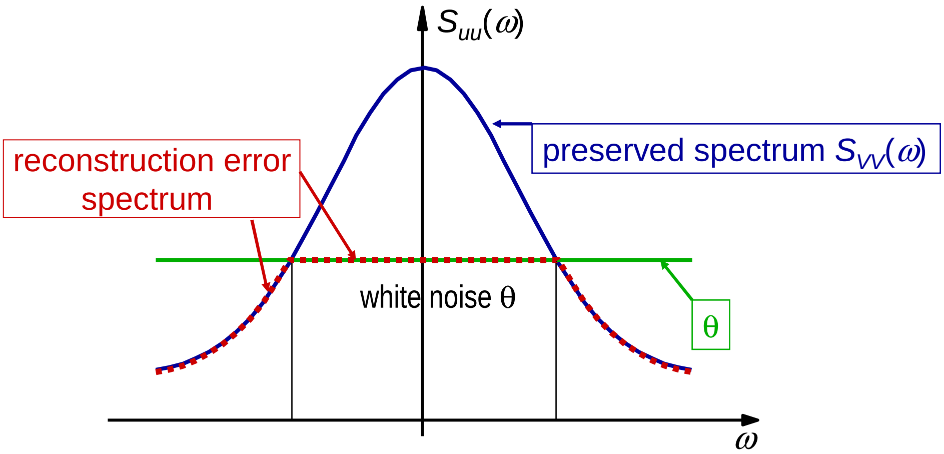
- Jointly Gaussian source with power spectrum $S_{uu}(\omega)$
- MSE: $D = E\{(u-v)^2\}$
- Parametric formulation of the $R(D^*)$ function

$$D = \frac{1}{2\pi}\int_{\omega}\min[\theta, S_{uu}(\omega)]d\omega$$

$$R = \frac{1}{2\pi}\int_{\omega}\max[0, \frac{1}{2}\log\frac{S_{uu}(\omega)}{\theta}]d\omega$$

- $R(D^*)$ for non-Gaussian sources with the same power spectral density is always lower.

## $R(D^*)$ Function for Gaussian Source with Memory II






## $R(D^*)$ Function for Gaussian Source with Memory III

- ACF and PSD for a first order AR(1) Gauss-Markov process: $U[n] = W[n] + \rho U[n-1]$

$$R_{uu}(k) = \rho^{|k|}\sigma^2, \qquad S_{uu}(\omega) = \frac{\sigma^2(1-\rho^2)}{1-2\rho\cos\omega+\rho^2}$$

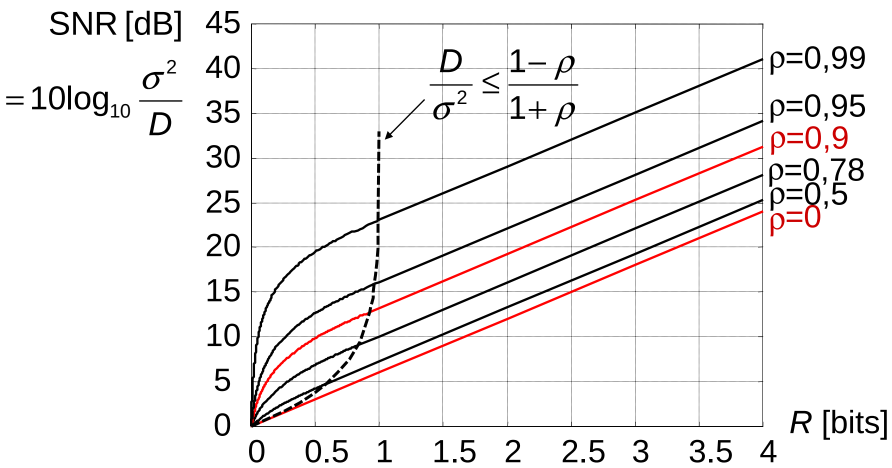
- Rate Distortion Function:

$$R(D^*) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\log_2\frac{S_{uu}(\omega)}{D^*}d\omega, \qquad \frac{D^*}{\sigma^2} \le \frac{1-\rho}{1+\rho}$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi}\log_2\frac{\sigma^2(1-\rho^2)}{D^*}d\omega - \frac{1}{4\pi}\int_{-\pi}^{\pi}\log_2\left(1-2\rho\cos\omega+\rho^2\right)d\omega$$

$$= \frac{1}{2}\log_2\frac{\sigma^2\left(1-\rho^2\right)}{D^*} = \frac{1}{2}\log_2\frac{\sigma_z^{\ 2}}{D^*}.$$

## $R(D^*)$ Function for Gaussian Source with Memory IV






## Quantization

- Structure

$$u \longrightarrow \boxed{\text{Quantizer}} \longrightarrow v$$

- Alternative: coder ($\alpha$) / decoder ($\beta$) structure

$$u \longrightarrow \boxed{\alpha} \xrightarrow{i} \boxed{\beta} \xrightarrow{v}$$

- Insert entropy coding ($\gamma$) and transmission channel

$$u \longrightarrow \boxed{\alpha} \xrightarrow{i} \boxed{\gamma} \xrightarrow{b} \boxed{\textit{channel}} \xrightarrow{b} \boxed{\gamma^{-1}} \xrightarrow{i} \boxed{\beta} \xrightarrow{v}$$

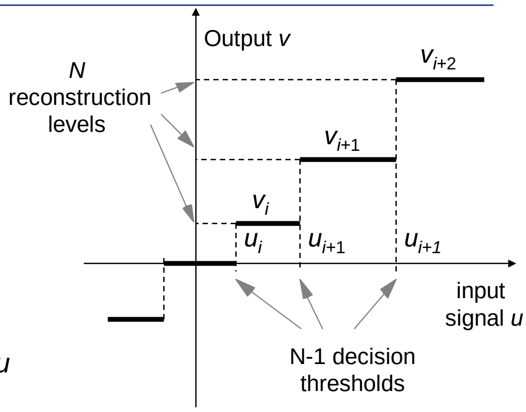
# Scalar Quantization

- Average distortion

$$D = E\{d(U,V)\}$$
$$= \sum_{k=0}^{N-1} \int_{u_k}^{u_{k+1}} d(u,v_k) \cdot f_U(u) \mathrm{d}u$$

- Assume MSE

$$d(u,v_k) = (u - v_k)^2$$
$$D = \sum_{k=0}^{N-1} \int_{u_k}^{u_{k+1}} (u - v_k)^2 \cdot f_U(u) \mathrm{d}u$$




- Fixed code word length vs. variable code word length

$$R = \log N \quad vs. \quad R = -E\{\log P(v)\}$$



# Lloyd-Max Quantizer

0: Given: a source distribution $f_U(u)$
a set of reconstruction levels $\{\beta_k\}$

1: Encode given $\{\beta_k\}$ (Nearest Neighbor Condition):
$\alpha(u) = \text{argmin}\ \{d(u,\beta_k)\}$ → $u_k=(v_k+v_{k+1})/2$ (MSE)

2: Update set of reconstruction levels given $\alpha(u_k)$
(Centroid Condition):

$$\beta_k = \text{argmin}\ E\ \{\ d(u,\beta_k)\ |\ \alpha(u)=k\} \rightarrow v_k = \frac{\int_{u_k}^{u_{k+1}} u \cdot f_U(u) \mathrm{d}u}{\int_{u_k}^{u_{k+1}} f_U(u) \mathrm{d}u} \quad \text{(MSE)}$$

3: Repeat steps 1 and 2 until convergence

## High Resolution Approximations

- Pdf of $U$ is roughly constant over individual cells $C_k$

$$f_U(u) \approx f_k, \quad u \in C_k$$

- The fundamental theorem of calculus

$$P_k = \Pr(u \in C_k) = \int_{u_k}^{u_{k+1}} f_U(u) \cdot \mathrm{d}u \approx (u_{k+1} - u_k) \cdot f_k = \Delta_k f_k$$

- Approximate average distortion (MSE)

$$D = \sum_{k=0}^{N-1} \int_{u_k}^{u_{k+1}} (u - v_k)^2 \cdot f_U(u) \cdot \mathrm{d}u = \sum_{k=0}^{N-1} f_k \int_{u_k}^{u_{k+1}} (u - v_k)^2 \mathrm{d}u$$

$$= \sum_{k=0}^{N-1} f_k \frac{\Delta_k^3}{12} = \frac{1}{12} \sum_{k=0}^{N-1} P_k \Delta_k^2$$

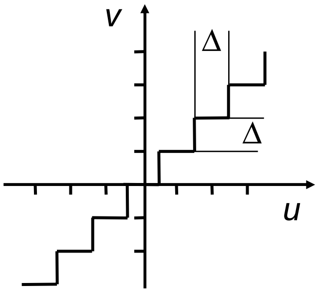


## Uniform Quantization

- Reconstruction levels of quantizer $\{v_k\}, k \in K$ are uniformly spaced
- Quantizer step size, i.e. distance between reconstruction levels: $\Delta$
- Average distortion

$$\sum_{k=0}^{N-1} P_k = 1, \quad \Delta_k = \Delta$$

$$D = \frac{1}{12} \sum_{k=0}^{N-1} P_k \Delta_k^2 = \frac{\Delta^2}{12} \sum_{k=0}^{N-1} P_k = \frac{\Delta^2}{12}$$




- Closed-form solutions for pdf-optimized uniform quantizers for Gaussian RV only exist for $N$=2 and $N$=3
- Optimization of $\Delta$ is conducted numerically

## Panter and Dite Approximation

- Approximate solution for optimized spacing of reconstruction and decision levels
- Assumptions: high resolution and smooth pdf $\Delta(u)$

$$\Delta(u) = \frac{\text{const}}{\sqrt[3]{f_U(u)}}$$

- Optimal pdf of reconstruction levels is not the same as for the input levels
- Average Distortion $\quad D \approx \frac{1}{12N^2}\left(\int_{\Re} f_U^{1/3}(u)\cdot \mathrm{d}u\right)^3$
- Operational distortion rate function for Gaussian RV

$$U \sim \mathcal{N}(0,\sigma^2), \quad D(R) \approx \frac{\pi\sqrt{3}}{2}\sigma^2 2^{-2R}$$

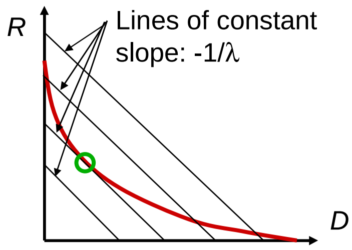


## Entropy-Constrained Quantization

- So far: each reconstruction level is transmitted with fixed code word length
- Encode reconstruction levels with variable code word length
- Constrained design criteria:

  min $D$, s.t. $R < R_c$ or min $R$, s.t. $D < D_c$
- Pose as unconstrained optimization via Lagrangian formulation:

  min $D + \lambda R$




- For a given $\lambda$, an optimum is obtained corresponding to either $R_c$ or $D_c$
- If $\lambda$ small, then $D$ small and $R$ large
  if $\lambda$ large, then $D$ large and $R$ small
- Optimality also for functions that are neither continuous nor differentiable

## Chou, Lookabaugh, and Gray Algorithm*

0: Given: a source distribution $f_U(u)$
a set of reconstruction levels $\{\beta_k\}$
a set of variable length code (VLC) words $\{\gamma_k\}$
set $n$=1

1: Encode given $\{\beta_k\}$ and $\{\gamma_k\}$:
$\alpha(u) = \operatorname{argmin}\,\{d(u,\beta_k)\}+\lambda|\gamma_k|$

2: Update VLC given $\alpha(u_k)$ and $\{\beta_k\}$
$|\gamma_k| = -\log P(\alpha(u)=k)$

3: Update set of reconstruction levels given $\alpha(u_k)$ and $\{\gamma_k\}$
$\beta_k = \operatorname{argmin}\, E\,\{\, d(u,\beta_k) \mid \alpha(u)=k\}$

4: Repeat steps 1 - 3 until convergence

*1989, has been proposed for Vector Quantization



## Entropy-Constrained Scalar Quantization: High Resolution Approximations

- Assume: uniform quantization: $P_k = f_k\,\Delta$

$$\begin{aligned} R &= -\sum_{k=0}^{N-1} P_k \log P_k = -\sum_{k=0}^{N-1} f_k \Delta \log(f_k \Delta) \\ &= -\sum_{k=0}^{N-1} f_k \Delta \log(f_k) - \sum_{k=0}^{N-1} f_k \Delta \log(\Delta) \\ &\approx \underbrace{\int_{\Re} f_U(u) \log(f_U(u))\mathrm{d}u}_{\text{Differential Entropy } h(U)} - \log\Delta \underbrace{\int_{\Re} f_U(u)\mathrm{d}u}_{1} \\ &= h(U) - \log\Delta \end{aligned}$$

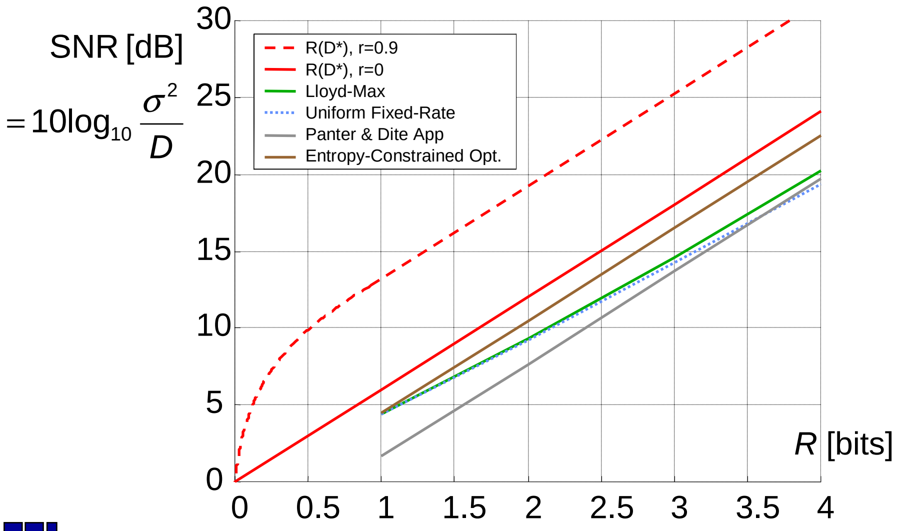
$$\sum \Delta \approx \int \mathrm{d}u$$

- Operational distortion rate function for Gaussian RV

$$U \sim N(0,\sigma^2), \quad D(R) \approx \frac{\pi\,\mathrm{e}}{6}\sigma^2 2^{-2R}$$

- It can be shown that for high resolution:
*Uniform Entropy-Constrained Scalar Quantization is optimum*

# Comparison for Gaussian Sources

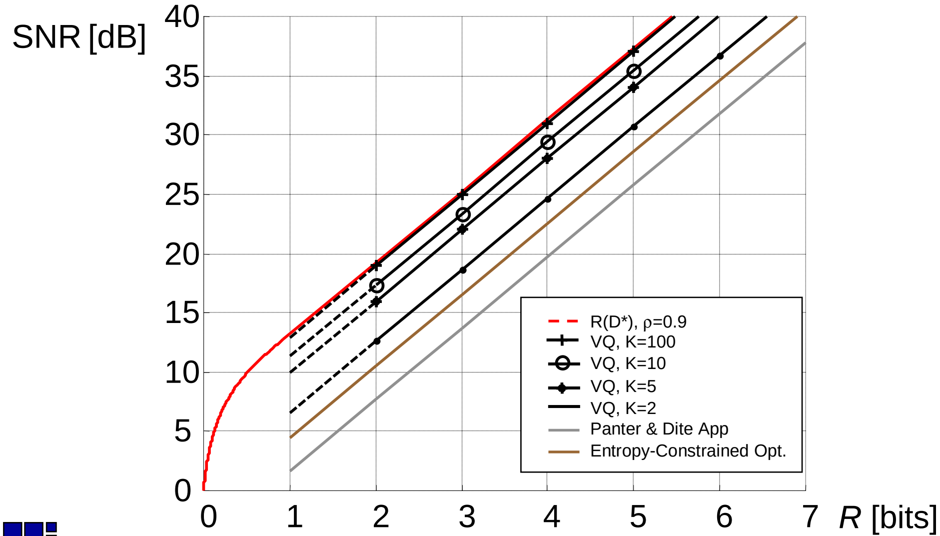





# Vector Quantization

- So far: scalars have been quantized
- Encode vectors, ordered sets of scalars
- Gain over scalar quantization (Lookabaugh and Gray 1989)
  - Space filling advantage
    - Z lattice is not most efficient sphere packing in *K-D* (*K*>1)
    - Independent from source distribution or statistical dependencies
    - Maximum gain for $K \rightarrow \infty$: 1.53 dB
  - Shape advantage
    - Exploit shape of source pdf
    - Can also be exploited using entropy-constrained scalar quantization
  - Memory advantage
    - Exploit statistical dependencies of the source
    - Can also be exploited using DPCM, Transform coding, block entropy coding

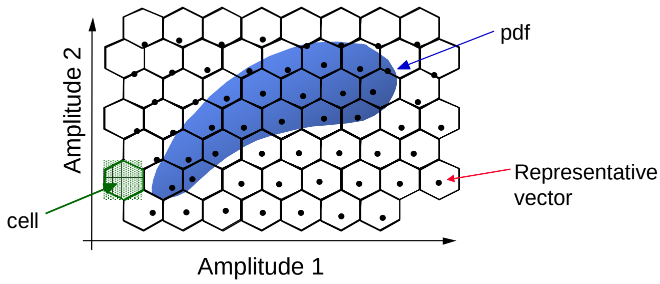
## Comparison for Gauss-Markov Source: ρ=0.9






# Vector Quantization II

- Vector quantizers can achieve $R(D^*)$ if $K \rightarrow \infty$
- Complexity requirements: storage and computation
- Delay
- Impose structural constraints that reduce complexity
- Tree-Structured, Transform, Multistage, etc.
- Lattice Codebook VQ



# Summary

- Rate-distortion theory: minimum bit-rate for given distortion
- $R(D^*)$ for memoryless Gaussian source and MSE: 6 dB/bit
- $R(D^*)$ for Gaussian source with memory and MSE: encode spectral components independently, introduce white noise, suppress small spectral components
- Lloyd-Max quantizer: minimum MSE distortion for given number of representative levels
- Variable length coding: additional gains by entropy-constrained quantization
- Minimum mean squared error for given entropy: uniform quantizer (for fine quantization!)
- Vector quantizers can achieve $R(D^*)$ if $K \rightarrow \infty$ : Are we done ?
- No! Complexity of vector quantizers is the issue

Design a coding system with optimum rate distortion performance, such that the delay, complexity, and storage requirements are met.