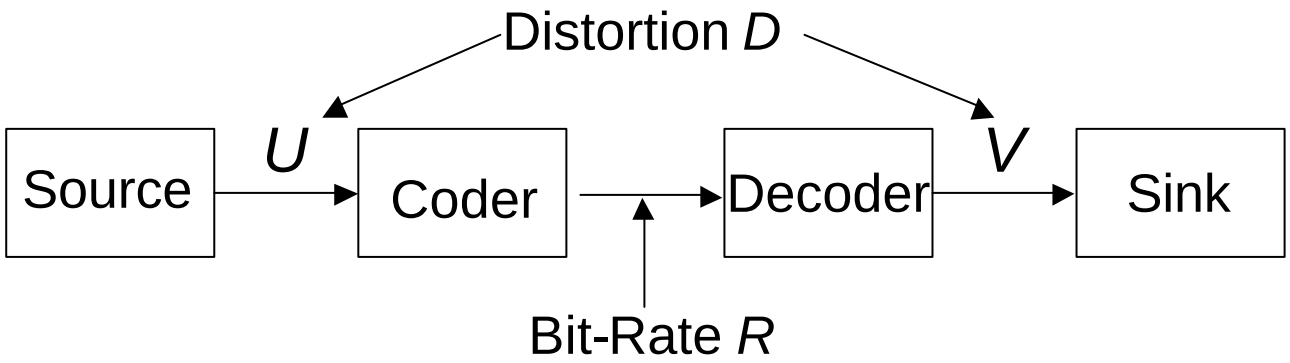
Rate Distortion Theory & Quantization

- Rate Distortion Theory
- Rate Distortion Function
- $R(D^*)$ for Memoryless Gaussian Sources
- $R(D^*)$ for Gaussian Sources with Memory
- Scalar Quantization
- Lloyd-Max Quantizer
- High Resolution Approximations
- Entropy-Constrained Quantization
- Vector Quantization

Rate Distortion Theory

- Theoretical discipline treating data compression from the viewpoint of information theory.
- Rate distortion theory calculates minimum transmission bit-rate *R* for a given distortion *D*.



- Results of rate distortion theory are obtained without consideration of a specific coding method.

Rate Distortion Theory: Mutual Information

- "Mutual information" is the information that symbols *u* and symbols *v* convey about each other.

$$I(u_l;v_k) = I(v_k) - I(v_k|u_l) = \log\frac{P(v_k|u_l)}{P(v_k)} = \log\frac{P(u_l|v_k)}{P(u_l)}$$

- Average mutual information:

$$I(U;V) = \sum_u \sum_v P(u,v)\cdot\log\frac{P(v|u)}{P(v)} = \sum_u \sum_v P(u,v)\cdot\log\frac{P(u|v)}{P(u)}$$
$$= H(V) - H(V|U) = H(U) - H(U|V)$$

- Properties of mutual information:

$$0 \le I(U;V) = I(V;U)$$
$$I(U;V) \le H(U)$$
$$I(V;U) \le H(V)$$

Rate Distortion Theory: Distortion

- Symbol u sent, v received
- Distortion

$$d(u,v) \geq 0$$
$$d(u,v) = 0 \quad for \quad u = v$$

- Average distortion:

$$D = E[d(u,v)] = \sum_u \sum_v P(v|u) \cdot P(u) \cdot d(u,v)$$

- Distortion criterion:

$$D \leq D^*$$

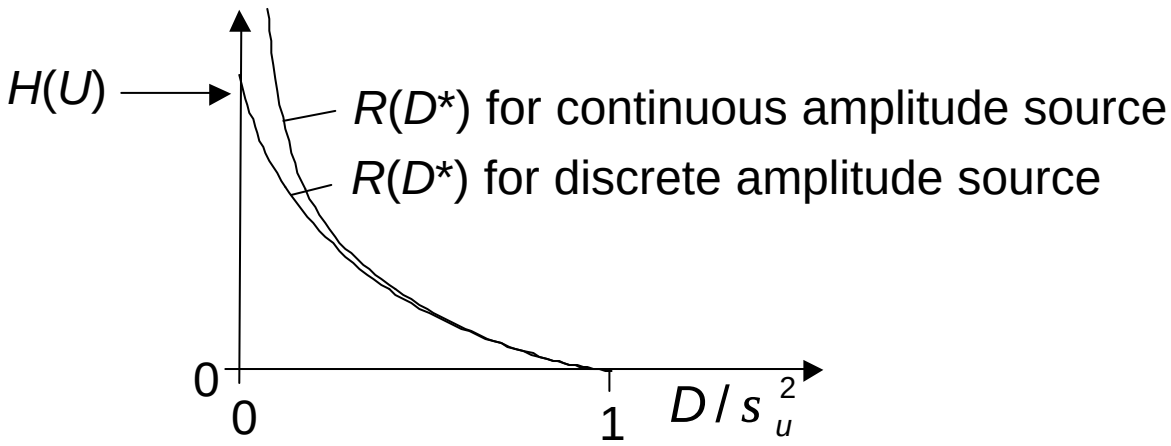
(D^* -maximum average distortion)

Rate Distortion Function

- Definition:

$$R(D^*) = \min_{D \le D^*} \{I(U;V)\}$$

- For a given maximum average distortion D^*, the rate distortion function $R(D^*)$ is the lower bound for the transmission bit-rate.



Shannon Lower Bound

- It can be shown that $$H(U-V|V)=H(U|V)$$

- Then we can write

$$\begin{aligned} R(D^*) &= \min_{D \le D^*} \{H(U)-H(U|V)\} \\ &= H(U) - \max_{D \le D^*} \{H(U|V)\} \\ &= H(U) - \max_{D \le D^*} \{H(U-V|V)\} \end{aligned}$$

- Ideally, the source coder would produce distortions *u-v* that are statistically independent from the reconstructed signal *v* (not always possible!).

- Shannon Lower Bound: $$R(D^*) \ge H(U) - \max_{D \le D^*} H(U-V)$$

$R(D^*)$ for a Memoryless Gaussian Source and MSE Distortion

- Gaussian source, variance s^2
- Mean squared error (MSE) $D = E\{(u-v)^2\}$

$$R(D^*) = \frac{1}{2}\log\frac{s^2}{D^*}; \quad D(R^*) = s^2 2^{-2R^*}, R \geq 0$$

$$SNR = 10 \cdot \log_{10}\frac{s^2}{D} = 10 \cdot \log_{10} 2^{-2R} \approx 6R \quad [dB]$$

- Rule of thumb: 6 dB ~ 1 bit
- The $R(D^*)$ for non-Gaussian sources with the same variance s^2 is always below this Gaussian $R(D^*)$ - curve.

$R(D^*)$ Function for Gaussian Source with Memory I

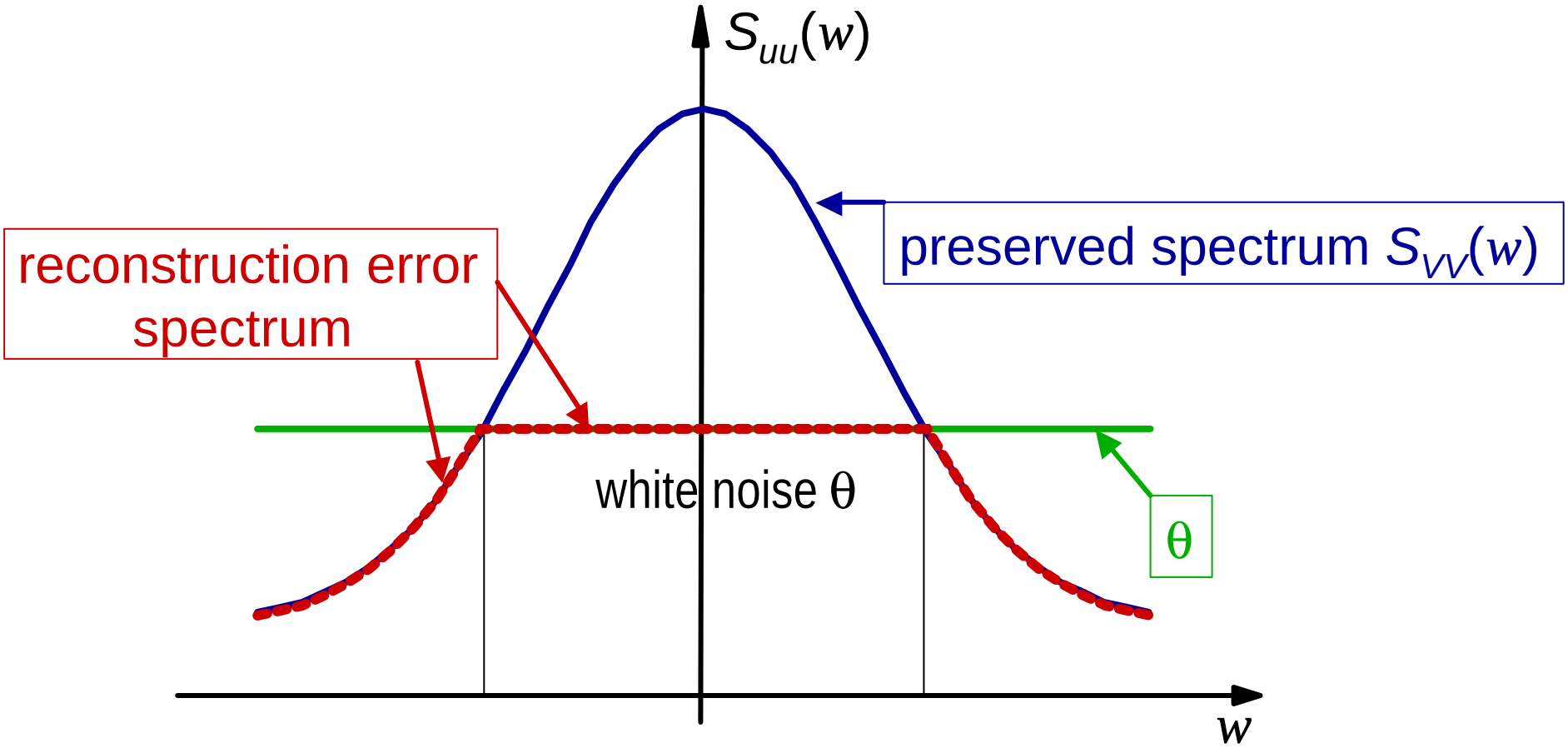
- Jointly Gaussian source with power spectrum $S_{uu}(w)$
- MSE: $D = E\{(u-v)^2\}$
- Parametric formulation of the $R(D^*)$ function

$$D = \frac{1}{2p} \int_w \min[q, S_{uu}(w)] dw$$

$$R = \frac{1}{2p} \int_w \max[0, \frac{1}{2} \log \frac{S_{uu}(w)}{q}] dw$$

- $R(D^*)$ for non-Gaussian sources with the same power spectral density is always lower.

$R(D^*)$ Function for Gaussian Source with Memory II



$R(D^*)$ Function for Gaussian Source with Memory III

- ACF and PSD for a first order AR(1) Gauss-Markov process: $U[n] = W[n] + \rho U[n-1]$

$$R_{uu}(k) = r^{|k|}s^2, \qquad S_{uu}(w) = \frac{s^2(1-r^2)}{1-2r\cos w + r^2}$$

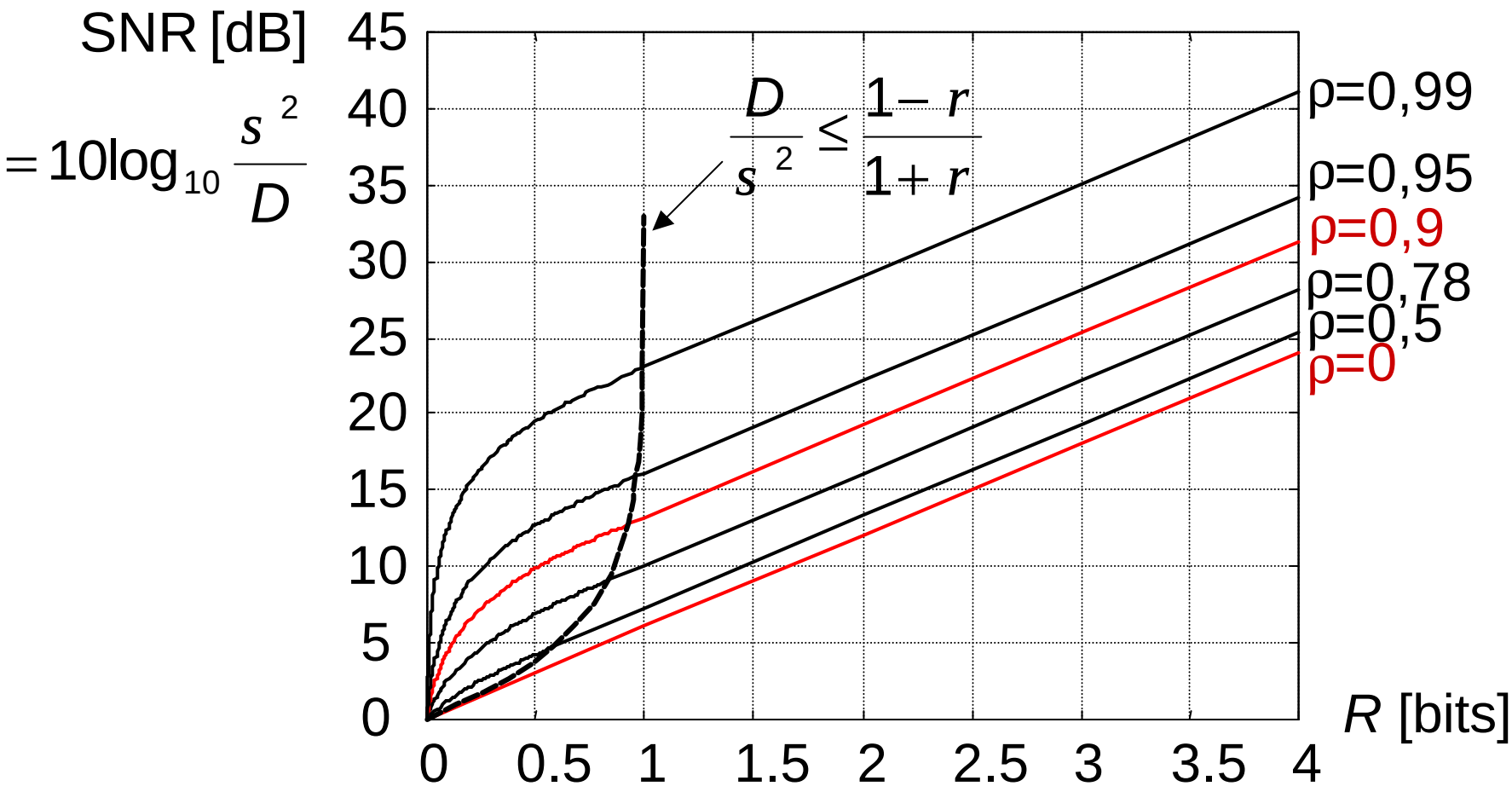
- Rate Distortion Function:

$$R(D^*) = \frac{1}{4p}\int_{-p}^{p}\log_2\frac{S_{uu}(w)}{D^*}dw, \qquad \frac{D^*}{s^2} \le \frac{1-r}{1+r}$$

$$= \frac{1}{4p}\int_{-p}^{p}\log_2\frac{s^2(1-r^2)}{D^*}dw - \frac{1}{4p}\int_{-p}^{p}\log_2\left(1-2r\cos w + r^2\right)dw$$

$$= \frac{1}{2}\log_2\frac{s^2\left(1-r^2\right)}{D^*} = \frac{1}{2}\log_2\frac{s_z^2}{D^*}.$$

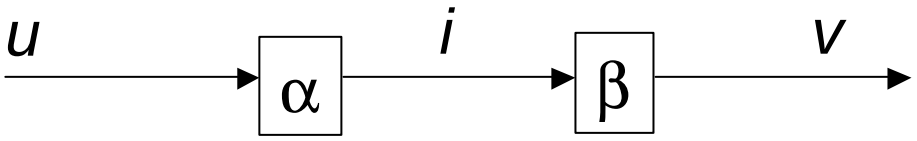
$R(D^*)$ Function for Gaussian Source with Memory IV



Quantization

- Structure

- Alternative: coder (α) / decoder (β) structure



- Insert entropy coding (γ) and transmission channel

Scalar Quantization

- Average distortion

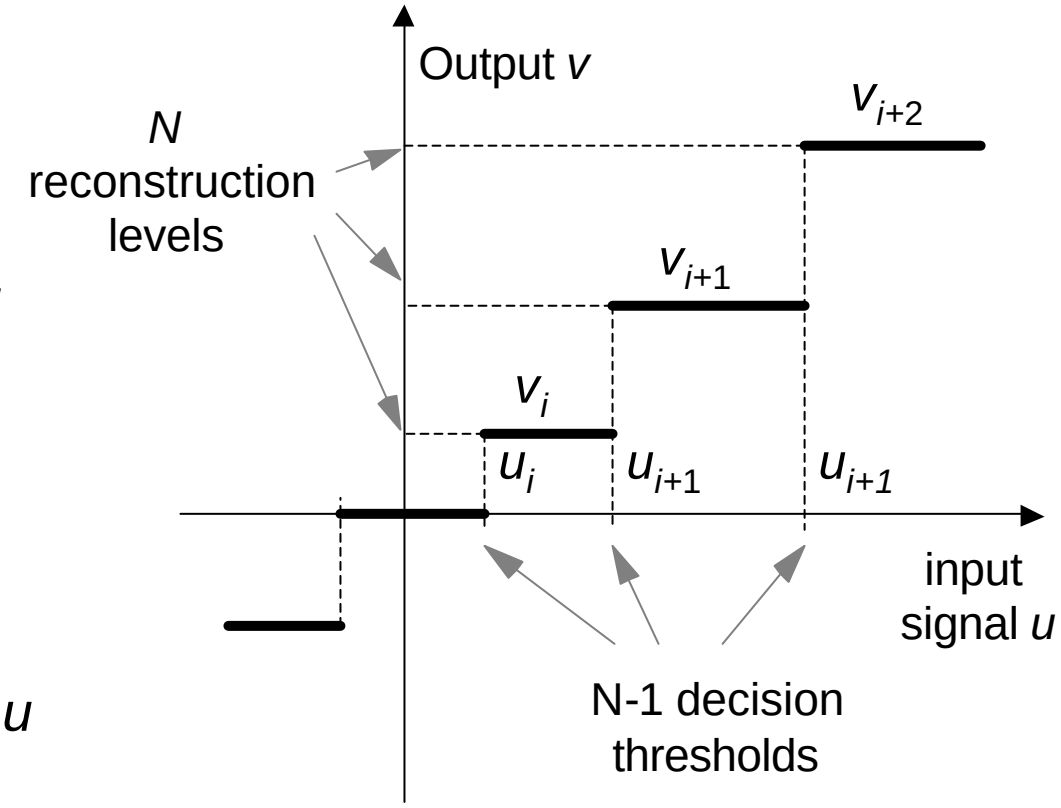
$$D = E\{d(U,V)\}$$

$$= \sum_{k=0}^{N-1} \int_{u_k}^{u_{k+1}} d(u,v_k) \cdot f_U(u)\mathrm{d}u$$

- Assume MSE

$$d(u,v_k) = (u - v_k)^2$$

$$D = \sum_{k=0}^{N-1} \int_{u_k}^{u_{k+1}} (u - v_k)^2 \cdot f_U(u)\mathrm{d}u$$



- Fixed code word length vs. variable code word length

$$R = \log N \quad vs. \quad R = -E\{\log P(v)\}$$

Lloyd-Max Quantizer

0: Given: a source distribution $f_U(u)$
a set of reconstruction levels $\{\beta_k\}$

1: Encode given $\{\beta_k\}$ (Nearest Neighbor Condition):

$\alpha(u) = \text{argmin}\ \{d(u,\beta_k)\}$ → $u_k=(v_k+v_{k+1})/2$ (MSE)

2: Update set of reconstruction levels given $\alpha(u_k)$ (Centroid Condition):

$$\beta_k = \text{argmin}\ E\{\, d(u,\beta_k) \mid \alpha(u)=k\} \rightarrow v_k = \frac{\int_{u_k}^{u_{k+1}} u \cdot f_U(u)\,\mathrm{d}u}{\int_{u_k}^{u_{k+1}} f_U(u)\,\mathrm{d}u} \quad \text{(MSE)}$$

3: Repeat steps 1 and 2 until convergence

High Resolution Approximations

- Pdf of U is roughly constant over individual cells C_k

$$f_U(u) \approx f_k, \quad u \in C_k$$

- The fundamental theorem of calculus

$$P_k = \Pr(u \in C_k) = \int_{u_k}^{u_{k+1}} f_U(u) \cdot \mathrm{d}u \approx (u_{k+1} - u_k) \cdot f_k = \Delta_k f_k$$

- Approximate average distortion (MSE)

$$D = \sum_{k=0}^{N-1} \int_{u_k}^{u_{k+1}} (u - v_k)^2 \cdot f_U(u) \cdot \mathrm{d}u = \sum_{k=0}^{N-1} f_k \int_{u_k}^{u_{k+1}} (u - v_k)^2 \mathrm{d}u$$

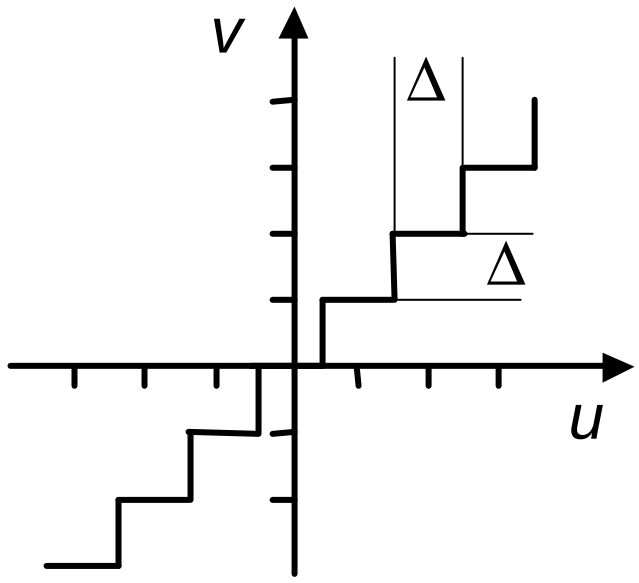
$$= \sum_{k=0}^{N-1} f_k \frac{\Delta_k^3}{12} = \frac{1}{12} \sum_{k=0}^{N-1} P_k \Delta_k^2$$

Uniform Quantization

- Reconstruction levels of quantizer $\{v_k\}, k \in K$ are uniformly spaced
- Quantizer step size, i.e. distance between reconstruction levels: Δ
- Average distortion

$$\sum_{k=0}^{N-1} P_k = 1, \quad \Delta_k = \Delta$$

$$D = \frac{1}{12}\sum_{k=0}^{N-1} P_k \Delta_k^2 = \frac{\Delta^2}{12}\sum_{k=0}^{N-1} P_k = \frac{\Delta^2}{12}$$



- Closed-form solutions for pdf-optimized uniform quantizers for Gaussian RV only exist for N=2 and N=3
- Optimization of Δ is conducted numerically

Panter and Dite Approximation

- Approximate solution for optimized spacing of reconstruction and decision levels
- Assumptions: high resolution and smooth pdf $\Delta(u)$

$$\Delta(u) = \frac{\text{const}}{\sqrt[3]{f_U(u)}}$$

- Optimal pdf of reconstruction levels is not the same as for the input levels
- Average Distortion $$D \approx \frac{1}{12N^2}\left(\int_{\Re} f_U^{1/3}(u)\cdot \mathrm{d}u\right)^3$$

- Operational distortion rate function for Gaussian RV

$$U \sim N(0, s^2), \quad D(R) \approx \frac{p\sqrt{3}}{2} s^2 2^{-2R}$$

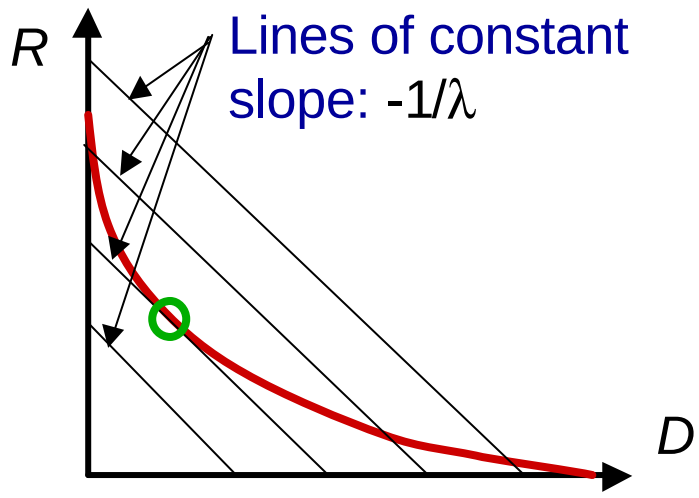
Entropy-Constrained Quantization

- So far: each reconstruction level is transmitted with fixed code word length
- Encode reconstruction levels with variable code word length
- Constrained design criteria:

$$\min D, \text{ s.t. } R < R_c \quad \text{or} \quad \min R, \text{ s.t. } D < D_c$$

- Pose as unconstrained optimization via Lagrangian formulation:

$$\min D + \lambda R$$



- For a given λ, an optimum is obtained corresponding to either R_c or D_c
- If λ small, then D small and R large
  if λ large, then D large and R small
- Optimality also for functions that are neither continuous nor differentiable

Chou, Lookabaugh, and Gray Algorithm*

0: Given: a source distribution $f_U(u)$
a set of reconstruction levels $\{\beta_k\}$
a set of variable length code (VLC) words $\{\gamma_k\}$
set n=1

1: Encode given $\{\beta_k\}$ and $\{\gamma_k\}$:
$$\alpha(u) = \text{argmin}\ \{d(u,\beta_k)\}+\lambda|\gamma_k|$$

2: Update VLC given $\alpha(u_k)$ and $\{\beta_k\}$
$$|\gamma_k| = -\log P(\alpha(u)=k)$$

3: Update set of reconstruction levels given $\alpha(u_k)$ and $\{\gamma_k\}$
$$\beta_k = \text{argmin}\ E\ \{\ d(u,\beta_k)\ |\ \alpha(u)=k\}$$

4: Repeat steps 1 - 3 until convergence

*1989, has been proposed for Vector Quantization

Entropy-Constrained Scalar Quantization: High Resolution Approximations

- Assume: uniform quantization: $P_k = f_k \Delta$

$$R = -\sum_{k=0}^{N-1} P_k \log P_k = -\sum_{k=0}^{N-1} f_k \Delta \log(f_k \Delta)$$

$$\sum \Delta \approx \int \mathrm{d}u \quad \downarrow$$

$$= -\sum_{k=0}^{N-1} f_k \Delta \log(f_k) - \sum_{k=0}^{N-1} f_k \Delta \log(\Delta)$$

$$\approx \underbrace{\int_{\Re} f_U(u) \log(f_U(u)) \mathrm{d}u}_{\text{Differential Entropy } h(U)} - \log \Delta \underbrace{\int_{\Re} f_U(u) \mathrm{d}u}_{1}$$

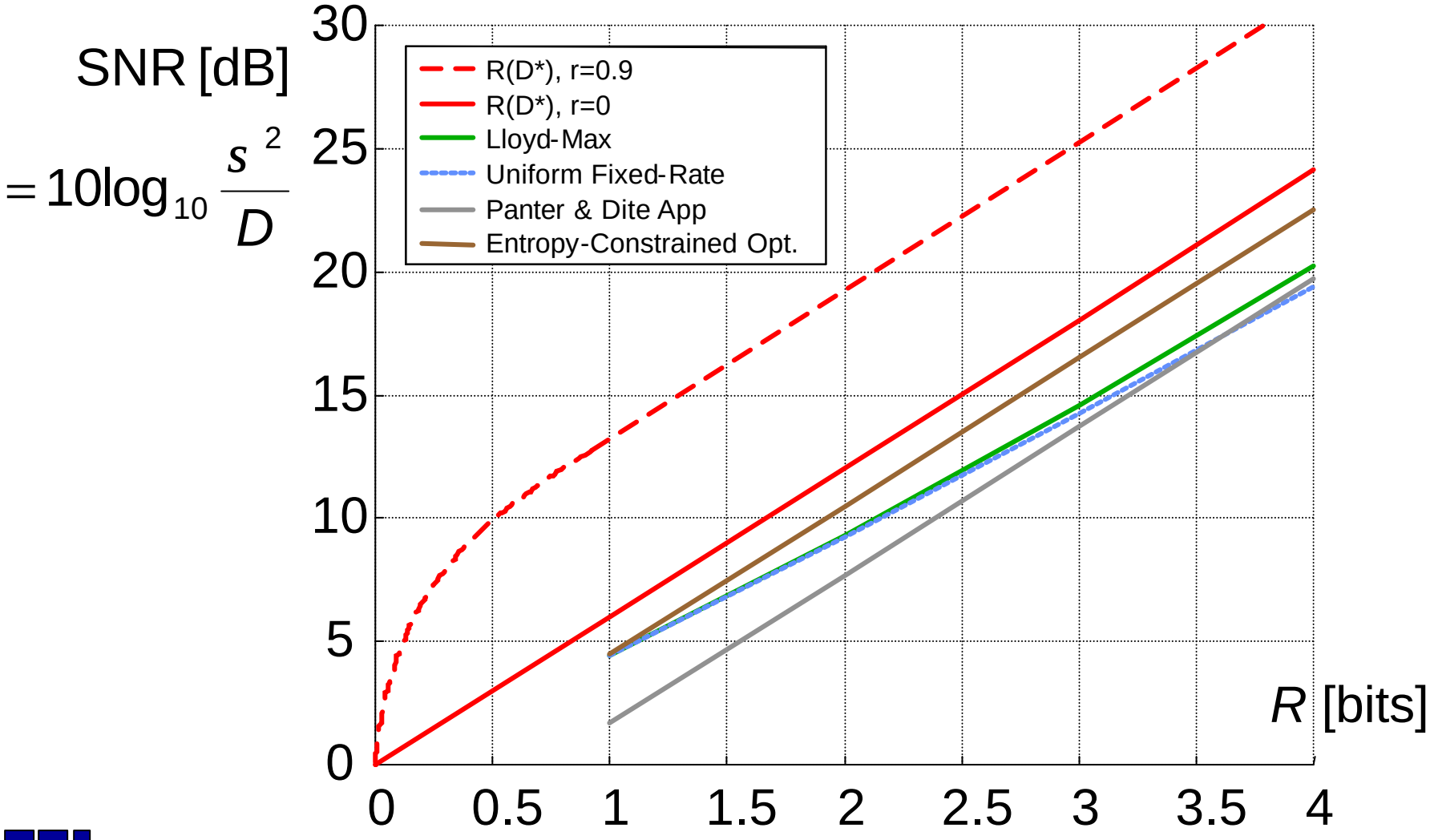
$$= h(U) - \log \Delta$$

- Operational distortion rate function for Gaussian RV

$$U \sim N(0, s^2), \quad D(R) \approx \frac{p\,\mathrm{e}}{6} s^2 2^{-2R}$$

- It can be shown that for high resolution:
  Uniform Entropy-Constrained Scalar Quantization is optimum

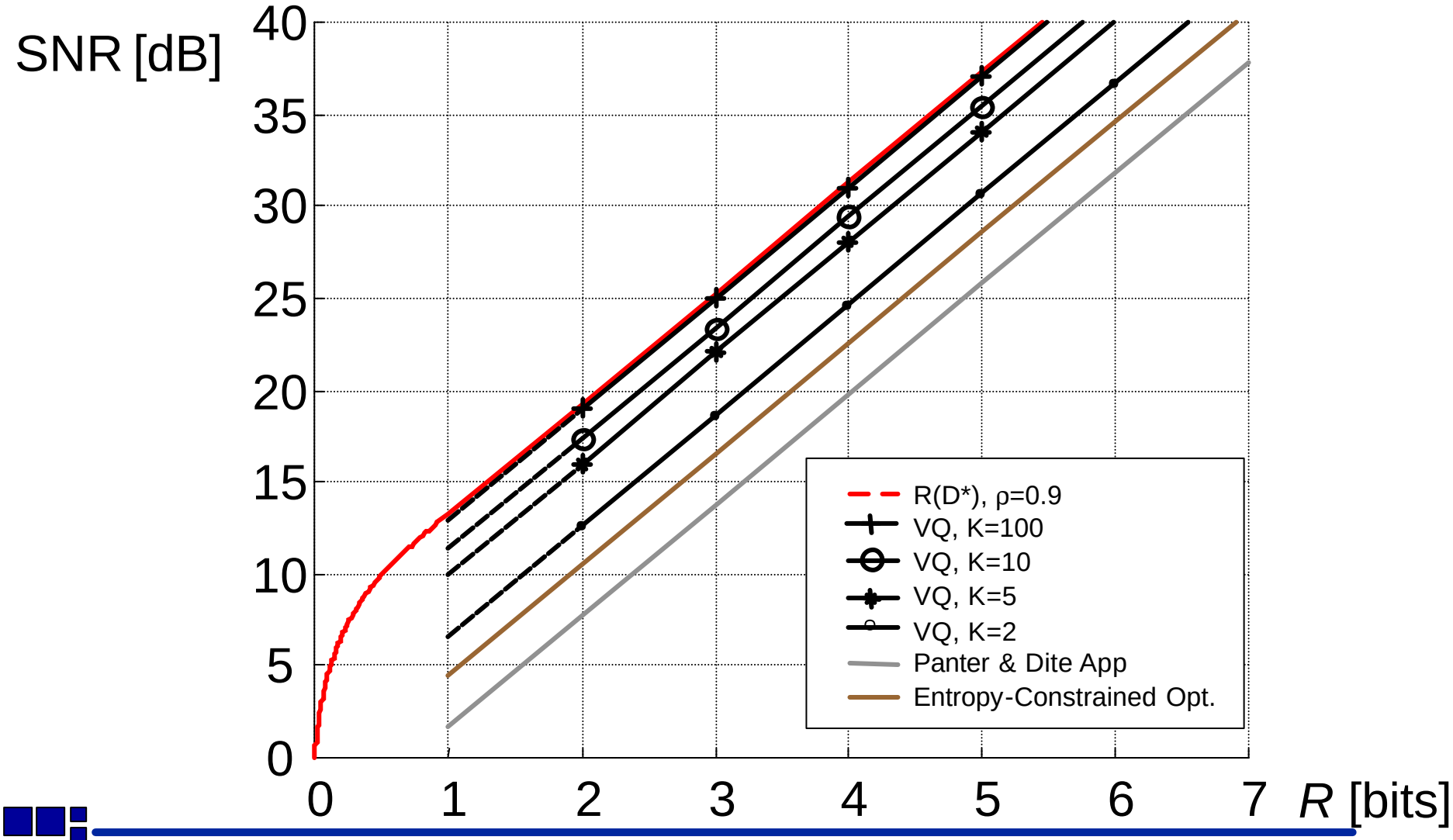
Comparison for Gaussian Sources



Vector Quantization

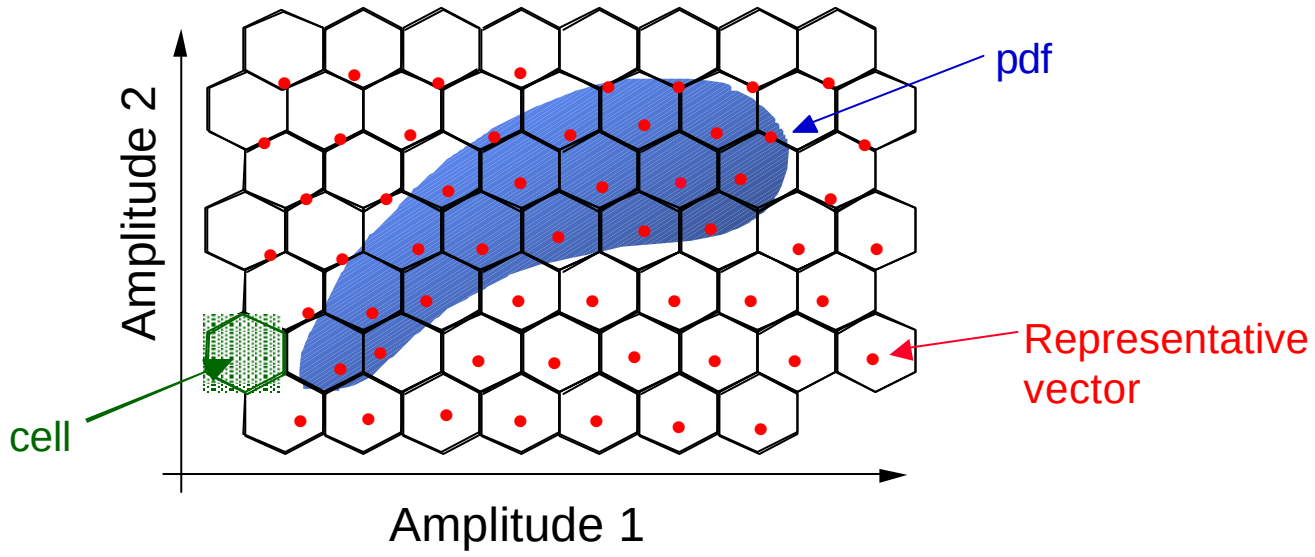
- So far: scalars have been quantized
- Encode vectors, ordered sets of scalars
- Gain over scalar quantization (Lookabaugh and Gray 1989)
  - ✓ Space filling advantage
    - Z lattice is not most efficient sphere packing in *K-D* (*K*>1)
    - Independent from source distribution or statistical dependencies
    - Maximum gain for $K \rightarrow \infty$: 1.53 dB
  - ✓ Shape advantage
    - Exploit shape of source pdf
    - Can also be exploited using entropy-constrained scalar quantization
  - ✓ Memory advantage
    - Exploit statistical dependencies of the source
    - Can also be exploited using DPCM, Transform coding, block entropy coding

Comparison for Gauss-Markov Source: ρ=0.9



Vector Quantization II

- Vector quantizers can achieve $R(D^*)$ if $K \rightarrow \infty$
- Complexity requirements: storage and computation
- Delay
- Impose structural constraints that reduce complexity
- Tree-Structured, Transform, Multistage, etc.
- Lattice Codebook VQ



Summary

- Rate-distortion theory: minimum bit-rate for given distortion
- $R(D^*)$ for memoryless Gaussian source and MSE: 6 dB/bit
- $R(D^*)$ for Gaussian source with memory and MSE: encode spectral components independently, introduce white noise, suppress small spectral components
- Lloyd-Max quantizer: minimum MSE distortion for given number of representative levels
- Variable length coding: additional gains by entropy-constrained quantization
- Minimum mean squared error for given entropy: uniform quantizer (for fine quantization!)
- Vector quantizers can achieve $R(D^*)$ if $K \rightarrow \infty$
- Complexity of vector quantizers

Design a coding system with optimum rate distortion performance, such that the delay, complexity, and storage requirements are met.