Information and Entropy

- Shannon’s Separation Principle
- Source Coding Principles
- Entropy
- Variable Length Codes
- Huffman Codes
- Joint Sources
- Arithmetic Codes
- Adaptive Codes

Shannon's Separation Principle

Assumptions:
- Single source and user
- Unlimited complexity and delay

Information Source	Generates information we want to transmit or store
Source Coding	Reduces number of bits to store or transmit relevant information
Channel Coding	Increases number of bits or changes them to protect against channel errors

Practical Systems

Many applications are not point-to-point transmissions:
• Consider feedback
• Networks

In any practical system, we cannot effort unlimited complexity neither unlimited delay:
• There will always be a small error rate unless we tolerate sub-optimality
• It might work better to consider source and channel coding jointly
• Consider effect of transmission errors on source decoding result

Source Coding Principles

• The source coder shall represent the video signal by the minimum number of (binary) symbols without exceeding an acceptable level of distortion.

• Two principles are utilized:

1. Properties of the information source that are known a priori result in redundant information that need not be transmitted (“redundancy reduction“).

2. The human observer does not perceive certain deviations of the received signal from the original (“irrelevancy reduction“).

• *Lossless* coding: completely reversible, exploit 1. principle only
• *Lossy* coding: not reversible, exploit 1. and 2. principle

from: Girod

Entropy of a Memoryless Source

- Let a memoryless source be characterized by an ensemble U_0 with:

Alphabet $\{ a_0, a_1, a_2, \ldots, a_{K-1} \}$

Probabilities $\{ P(a_0), P(a_1), P(a_2), \ldots, P(a_{K-1}) \}$

- Shannon: information conveyed by message "a_k":

$$I(a_k) = -\log(P(a_k))$$

- "Entropy of the source" is the <u>average</u> information contents:

$$H(U_0) = E\{I(a_k)\} = -\sum_{k=0}^{K-1} P(a_k) * \log(P(a_k))$$

- For „log" = „$\log_2$" the unit is bits/symbol

from: Girod

Entropy and Bit-Rate

- Properties of entropy:

$$H(U_0) \geq 0$$

$$\max\{ H(U_0) \} = \log K \text{ with } P(a_j) = P(a_k) \text{ for all } j, k$$

- The entropy $H(U_0)$ is a lower bound for the average word length l_{av} of a decodable variable length code
- Conversely, the average word length l_{av} can approach $H(U_0)$, if sufficiently large blocks of symbols are encoded jointly.
- Redundancy of a code: $r = l_{av} - H(U_0) \geq 0$

from: Girod

Encoding with Variable Word Lengths

- A code without redundancy, i.e.

$$\boxed{l_{av} = H(U_0)}$$

is achieved, if all individual code word lengths

$$\boxed{l_{cw}(a_k) = -\log(P(a_k))}$$

- For binary code words, all probabilities would have to be binary fractions:

$$\boxed{P(a_k) = 2^{-l_{cw}(a_k)}}$$

from: Girod

Redundant Codes: Example

a_i	$P(a_i)$	redundant code	optimum code
a_1	0.500	00	0
a_2	0.250	01	10
a_3	0.125	10	110
a_4	0.125	11	111
$H(U_0)$ =1.75 bits		l_{av}= 2 bits r = 0.25 bits	l_{av}=1.75 bits r = 0 bits

Variable Length Codes

- Unique decodabilty: Where does each code word start or end
- Insert start symbol: 01.0.010.1. wasteful
- Construct prefix-free code
- Kraft Inequality: test for uniquely decodable codes

Uniquely decodable code exists if $V = \sum_{k=0}^{K-1} 2^{-l_{cw}(a_k)} \le 1$

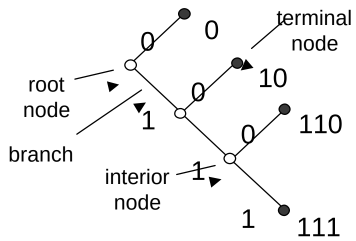
- Application:

a_i	$P(a_i)$	$-\log_2(P(a_i))$	Code A	Code B
a_1	0.5	1	0	0
a_2	0.2	2.32	01	10
a_3	0.2	2.32	10	110
a_4	0.1	3.32	111	111
			$V = 1.125$	$V = 1$

Not uniquely decodable | Uniquely decodable

Prefix-Free Codes

- Prefix-free codes are instantaneously and uniquely decodable
- Prefix-free codes can be represented by trees



- Terminal nodes may be assigned code words
- Interior nodes cannot be assigned code words
- For binary trees: *N* terminal nodes: *N-1* interior nodes

- Code 0, 01, 11 is not a prefix-free code and uniquely decodable but: non-instantaneous

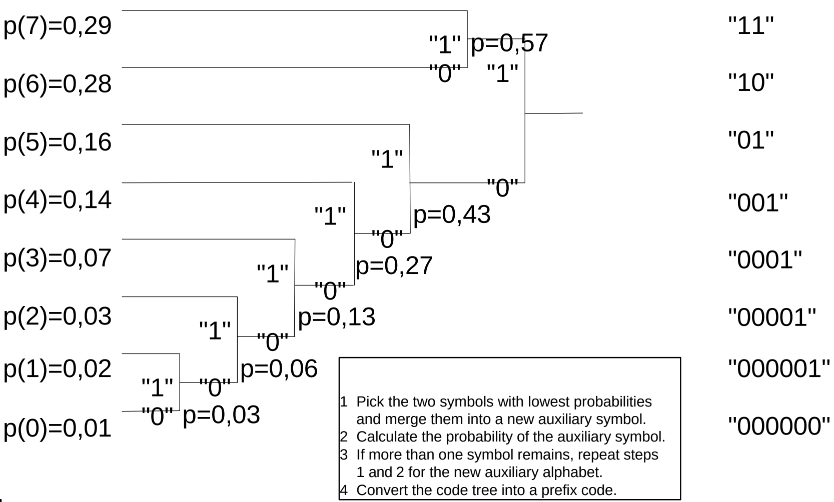
Huffman Code

- Design algorithm for variable length codes proposed by D. A. Huffman (1952) always finds a code with minimum redundancy.
- Obtain code tree as follows:

> 1 Pick the two symbols with lowest probabilities and merge them into a new auxiliary symbol.
> 2 Calculate the probability of the auxiliary symbol.
> 3 If more than one symbol remains, repeat steps 1 and 2 for the new auxiliary alphabet.
> 4 Convert the code tree into a prefix code.

from: Girod

Huffman Code: Example



from: Ohm

Joint Sources

- Joint sources generate N symbols simultaneously. A coding gain can be achieved by encoding those symbols jointly.
- The lower bound for the average code word length is the joint entropy:

$$H(U_1, U_2, \dots, U_N) = -\sum_{u_1} \sum_{u_2} \dots \sum_{u_N} P(u_1, u_2, \dots, u_N) \cdot \log(P(u_1, u_2, \dots, u_N))$$

- It generally holds that

$$H(U_1, U_2, \dots, U_N) \leq H(U_1) + H(U_2) + \dots + H(U_N)$$

with equality, if $U_1, U_2, \dots, U_N$ are statistically independent.

from: Girod

Markov Process

- Neighboring samples of the video signal are **not** statistically independent:

Source with memory

$$P(u_T) \neq P(u_T \mid u_{T-1}, u_{T-2}, \dots, u_{T-N})$$

- A source with memory can be modeled by a Markov random process.
- Conditional p robabilities of the source symbols u_T of a Markov source of order N:

$$P(u_T \mid Z_T) = P(u_T \mid u_{T-1}, u_{T-2}, \dots, u_{T-N})$$

state of the Markov source at time T

from: Girod

Entropy of Source with Memory

- Markov source of order N: conditional entropy

$$\begin{aligned} H(U_T \mid Z_T) &= H(U_T \mid U_{T-1}, U_{T-2}, \ldots, U_{T-N}) \\ &= -E\{\log(p(u_T \mid u_{T-1}, u_{T-2}, \ldots, u_{T-N}))\} \\ &= -\sum_{u_T} \ldots \sum_{u_{T-N}} p(u_T, u_{T-1}, u_{T-2}, \ldots, u_{T-N}) \log(p(u_T \mid u_{T-1}, u_{T-2}, \ldots, u_{T-N}) \end{aligned}$$

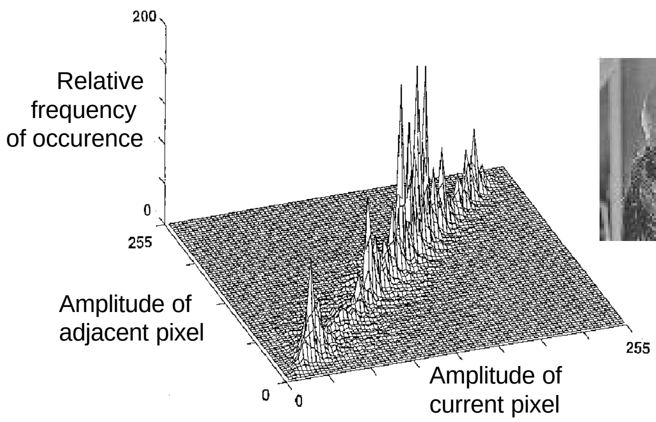
$$\boxed{H(U_T) \geq H(U_T \mid Z_T)}$$ (equality for memoryless source)

- Average code word length can approach $H(U_T \mid Z_T)$ e.g. with a switched Huffman code.
- Number of states for an 8-bit video signal:

$N = 1 \Rightarrow$ 256 states
$N = 2 \Rightarrow$ 65536 states
$N = 3 \Rightarrow$ 16777216 states

from: Girod

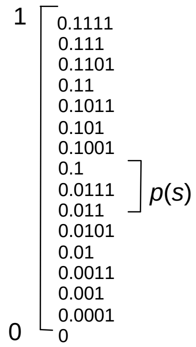
Second Order Statistics for Luminance Signal



Histogram of two horizontally adjacent pels
(picture: female head-and-shoulder view)

Arithmetic Coding

- Universal entropy coding algorithm for strings
- Representation of a string by a subinterval of the unit interval [0,1)
- Width of the subinterval is approximately equal to the probability of the string $p(s)$



- Interval of width $p(s)$ is guaranteed to contain one number that can be represented by b binary digits, with

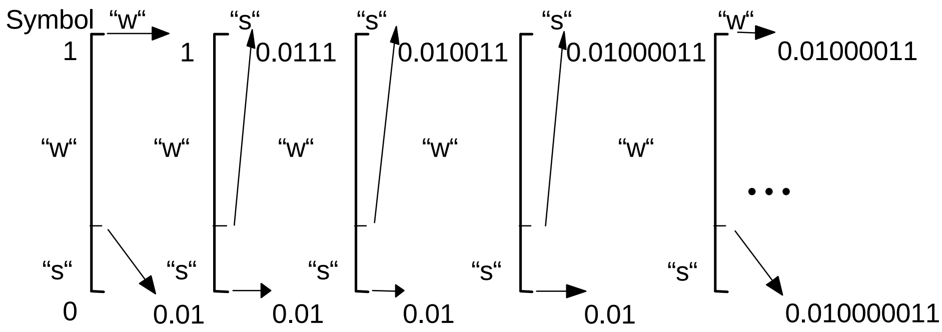
$$-\log(p(s)) + 1 \leq b \leq -\log(p(s)) + 2$$

- Each interval can be represented by a number which needs 1 to 2 bits more than the ideal code word length

from: Girod

Arithmetic Coding: Probability Intervals

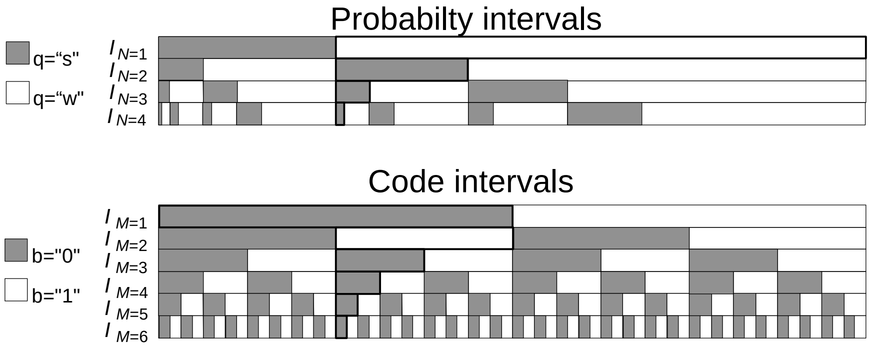
- Random experiment: pmf $p(\text{"s"}) = (0.01)_b$ and $p(\text{"w"}) = (0.11)_b$



- Multiplications and additions with (potentially) very long word length
- Universal coding: probabilities can be changed on the fly: e.g., use $p(\text{"s"} \mid \text{"s"})$, $p(\text{"s"} \mid \text{"w"})$, $p(\text{"w"} \mid \text{"s"})$, $p(\text{"w"} \mid \text{"w"})$

Arithmetic Encoding and Decoding

- Encoding: “w”, “s”, “s”, “s” ➔ 010000
- Decoding: 010 ➔ “w”, “s”

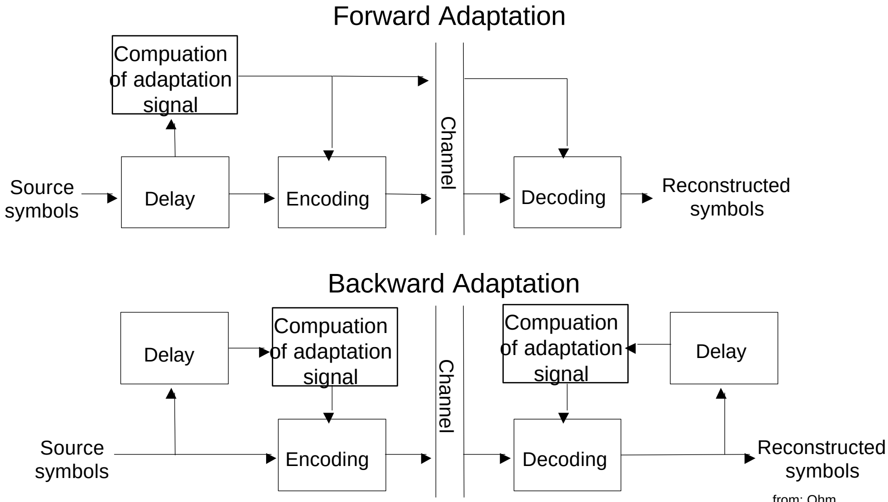


from: Ohm

Adaptive Entropy Coding

- For non-adaptive coding methods: pdf of source must be known a priori (inherent assumption: stationary source)
- Image and video signals are not stationary: sub-optimal performance
- Solution: adaptive entropy coding
- Two basic approaches to adaptation:
  1. Forward Adaptation
     - Gather statistics for a large enough block of source symbols
     - Transmit adaptation signal to decoder as side information
     - Drawback: increased bit-rate
  2. Backward Adaptation
     - Gather statistics simultaneously at coder and decoder
     - Drawback: error resilience
- Combine the two approaches and circumvent drawbacks (Packet based transmission systems)

Forward vs. Backward Adaptive Systems



from: Ohm

Summary

- Shannon’s information theory vs. practical systems
- Source coding principles: redundancy & irrelevancy reduction
- Lossless vs. lossy coding
- Redundancy reduction exploits the properties of the signal source.
- Entropy is the lower bound for the average code word length.
- Huffman code is optimum entropy code.
- Huffman coding: needs code table.
- Arithmetic coding is a universal method for encoding strings of symbols.
- Arithmetic coding does not need a code table.
- Adaptive entropy coding: gains for sources that are not stationary