Proofs for the Optimum Bit Allocation Rules

In this document we state the calculations which lead to the optimum bit allocation rules given by Eq. (2) for open-loop (OL) coding and Eq. (3) for closed-loop (CL) coding in

U. Horn, T. Wiegand, and B. Girod: "Bit Allocation Methods for Closed-Loop Coding of Oversampled Pyramid Decompositions", to be published in *Proceedings of the IEEE International Conference on Image Processing*, 26-29 October 1997, Santa Barbara, USA.

1 Optimum Bit Allocation for Open-Loop Coding

Allocating rates r_l to layers $0 \leq l \leq L-1$ and assuming a Gaussian source, the Eq. for the distortion in the open-loop case D_{OL} is given as

$$D_{OL}(r_l) = \sum_{l=0}^{L-1} \sigma_l' \cdot 2^{-2r_l} \tag{1}$$

with $\sigma_l' = \alpha_l \cdot g_l \cdot \sigma_l^2$, where α_l is a factor which represents the power transfer factor of the cascade of l interpolation filters, g_l expresses the spectral flatness, and σ_l^2 is the variance of the interpolation error signal in layer l. The bit allocation problem can be stated as

$$\min_{r_l} D_{OL}(r_l) = \min_{r_l} \left(\sum_{l=0}^{L-1} \sigma_l' \cdot 2^{-2r_l} \right) \quad \text{subject to} \quad \sum_{l=0}^{L-1} n_l \cdot r_l \;\; \leq \;\; R, \tag{2}$$

where we minimize the distortion D_{OL} subject to a rate-constraint R. $n_l = N_l/N_0$ is defined as the ratio between N_l the number of samples in layer l and N_0 the number of samples in the full resolution layer 0. By introducing a Lagrange multiplier λ, the constrained minimization problem (2) becomes an unconstrained one

$$\min_{r_l} J_{OL}(r_l) = \min_{r_l} \left(\sum_{l=0}^{L-1} \sigma_l' \cdot 2^{-2r_l} + \lambda \sum_{l=0}^{L-1} n_l \cdot r_l \right). \tag{3}$$

By assuming our variables being continuous $d(r_l), r_l \in \mathbb{R}$ and always differentiable, we obtain the necessary conditions for optimality by setting the partial derivates subject to r_l to zero

$$\frac{\partial J_{OL}(r_l)}{\partial r_l} = (-2) \cdot \ln 2 \cdot \sigma_l' \cdot 2^{-2r_l} + \lambda n_l \stackrel{!}{=} 0, \qquad \forall l: \; 0 < l < L-1 \tag{4}$$

From the necessary condition (4), the optimum bit allocation r_l for layer l can be written as

$$r_l = \frac{1}{2} \log_2 \left(\frac{\sigma_l'}{n_l} \frac{2 \cdot \ln 2}{\lambda} \right) = \frac{1}{2} \log_2 \frac{\sigma_l'}{n_l} - \frac{1}{2} \log_2 \frac{\lambda}{2 \cdot \ln 2} \tag{5}$$

Since our variables are continuous, we can change the inequality for the rate-constraint into an equality

$$R = \sum_{l=0}^{L-1} n_l \cdot r_l. \tag{6}$$

and eliminate the dependency of r_l on the Lagrange multiplier λ. For that, we plug (5) into (6) yielding

$$R = \sum_{k=0}^{L-1} n_k \cdot \left(\frac{1}{2} \log_2 \frac{\sigma_k'}{n_k} - \frac{1}{2} \log_2 \frac{\lambda}{2 \cdot \ln 2} \right) \tag{7}$$

The term containing λ is independent of the summing index k. Hence, (7) can be written

$$R = \sum_{k=0}^{L-1} n_k \cdot \frac{1}{2} \log_2 \frac{\sigma_k'}{n_k} - \frac{M}{2} \log_2 \frac{\lambda}{2 \cdot \ln 2}. \tag{8}$$

with $M = \sum_{k=0}^{L-1} n_k$ representing the redundancy of the oversampled decomposition. Eliminating the term that contains λ and dividing both sides by M we get

$$-\frac{1}{2} \log_2 \frac{\lambda}{2 \cdot \ln 2} = \frac{R}{M} - \frac{1}{M} \sum_{k=0}^{L-1} n_k \cdot \frac{1}{2} \log_2 \frac{\sigma_k'}{n_k}. \tag{9}$$

Plugging the left hand side of (9) into (5) yields

$$\begin{aligned} r_l &= \frac{1}{2} \log_2 \frac{\sigma_l'}{n_l} + \frac{R}{M} - \frac{1}{M} \sum_{k=0}^{L-1} n_k \cdot \frac{1}{2} \log_2 \frac{\sigma_k'}{n_k} \\ &= \frac{R}{M} + \frac{1}{2} \log_2 \frac{\sigma_l'}{n_l} - \sum_{k=0}^{L-1} \frac{1}{2} \log_2 \left(\frac{\sigma_k'}{n_k} \right)^{\frac{n_k}{M}} \end{aligned}$$

which we can recast as

$$r_l = \frac{R}{M} + \frac{1}{2} \log_2 \frac{\alpha_l \cdot g_l \cdot \sigma_l^2 / n_l}{\prod_{k=0}^{L-1} (\alpha_k \cdot g_k \cdot \sigma_k^2 / n_k)^{\frac{n_k}{M}}}, \tag{10}$$

remembering that $\sigma_l' = \alpha_l \cdot g_l \cdot \sigma_l^2$.

2 Optimum Bit Allocation for Closed-Loop Coding

In contrast to open-loop coding, the distortion at each layer l is dependent on the distortion introduced in layers $l+1, \cdots, L-1$ due to the noise-feedback. Thus, the overall distortion for closed-loop (CL) coding $D_{CL}(r_l)$ is given as

$$D_{CL}(r_l) = \sum_{l=0}^{L-1} \sigma_l' \prod_{k=0}^{l} 2^{-2r_k}, \tag{11}$$

In contrast to open-loop coding, we set $\sigma_l' = \alpha^l \cdot g_l \cdot \sigma_l^2$, where α is the power transfer factor of the interpolation filter. At this point, we assume that the *spectral flatness* values g_l are independent of the rates $r_{l+1}, \cdots, r_{L-1}$. Similar to open-loop coding we formulate the bit allocation problem as a unconstrained optimization problem

$$\min_{r_l} J_{CL}(r_l) = \min_{r_l} \left(\sum_{l=0}^{L-1} \sigma_l' \prod_{k=0}^{l} 2^{-2r_k} + \lambda \sum_{l=0}^{L-1} n_l \cdot r_l \right). \tag{12}$$

The necessary conditions for optimum bit allocation are

$$\frac{\partial J_{CL}(r_l)}{\partial r_l} = \sum_{i=l}^{L-1} \sigma_i' \prod_{k=0}^{i} (-2) \cdot \ln 2 \cdot 2^{-2r_k} + \lambda n_l \stackrel{!}{=} 0 \qquad \forall l:\ 0 \le l < L-1. \tag{13}$$

By letting the sum start at $i = l+1$, Eq. (13) can also be written as

$$\sigma_l' \prod_{k=0}^{l} (-2) \cdot \ln 2 \cdot 2^{-2r_k} + \sum_{i=l+1}^{L-1} \sigma_i' \prod_{k=0}^{i} (-2) \cdot \ln 2 \cdot 2^{-2r_k} + \lambda n_l = 0. \tag{14}$$

Substituting $l \to l+1$ in Eq. (13), the necessary condition for layer $l+1$ becomes

$$\sum_{i=l+1}^{L-1} \sigma_i' \prod_{k=0}^{i} (-2) \cdot \ln 2 \cdot 2^{-2r_k} + \lambda n_{l+1} = 0 \qquad \forall l:\ 0 \le l < L-2. \tag{15}$$

Subtracting (14) - (15) yields

$$\sigma_l' \prod_{k=0}^{l} \cdot 2^{-2r_k} = \frac{\lambda}{2 \cdot \ln 2}(n_l - n_{l+1}) \qquad \forall l:\ 0 \le l < L-2. \tag{16}$$

By substituting $l \to l-1$, Eq. (16) reads

$$\sigma_{l-1}' \prod_{k=0}^{l-1} \cdot 2^{-2r_k} = \frac{\lambda}{2 \cdot \ln 2}(n_{l-1} - n_l) \qquad \forall l:\ 1 < l < L-1. \tag{17}$$

Dividing Eq. (16) by Eq. (17) yields

$$\frac{\sigma_l'}{\sigma_{l-1}'} 2^{-2r_l} = \frac{n_l - n_{l+1}}{n_{l-1} - n_l} \qquad \forall l:\ 1 < l < L-2, \tag{18}$$

where $n_{l-1} - n_l \neq 0$ must hold. At this point, the Lagrange multiplier vanishes thus making bit allocation for layers $1 \ldots L-2$ (and for layer $L-1$ as we will show later) independent of the overall bit-rate R. Eliminating (18) for r_l leads us to

$$r_l \;=\; \frac{1}{2} \log_2 \left(\frac{\sigma_l'}{\sigma_{l-1}'} \cdot \frac{n_{l-1} - n_l}{n_l - n_{l+1}} \right) \qquad \forall l:\ 1 < l < L-2. \tag{19}$$

Note, that the argument of the logarithm should be a positive value, i.e.,

$$\frac{n_n - n_{l+1}}{n_{l-1} - n_l} > 0 \qquad \forall l:\ 1 < l < L-2. \tag{20}$$

Final questions are raised with regards to bit allocation in layers 0 and $L-1$. The latter, is easily answered by defining $n_L = 0$. With that, Eq. (19) is also valid for layer $L-1$. Having derived the bit allocation rules for layers $1 \ldots L-1$, the bit allocation rule for layer 0 is no more a free parameter, when having the rate-constraint given by Eq. (6). Hence, the optimum bit allocation rules for closed-loop coding can be written as

$$\begin{aligned} r_l &= \begin{cases} \frac{1}{2}\log_2\left(\frac{\alpha \cdot g_l \cdot \sigma_l^2}{g_{l-1} \cdot \sigma_{l-1}^2} \cdot w_l\right), & l > 0 \\ R - \sum_{l=1}^{L-1} n_l \cdot r_l, & l = 0 \end{cases} \\ w_l &= \begin{cases} \frac{n_{l-1} - n_l}{n_l - n_{l+1}}, & \frac{n_{l-1} - n_l}{n_l - n_{l+1}} > 0 \\ 1, & n_l = n_{l-1} = n_{l+1} \end{cases}, n_L = 0 \end{aligned} \tag{21}$$